

INERTIAL CHOW RINGS OF TORIC STACKS

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ABSTRACT. For any vector bundle V on a toric Deligne-Mumford stack \mathcal{X} the formalism of [EJK2] defines two inertial products \star_{V+} and \star_{V-} on the Chow group of the inertia stack. We give an explicit presentation for the integral \star_{V+} and \star_{V-} Chow rings, extending earlier work of Boris-Chen-Smith [BCS] and Jiang-Tsen [JT] in the orbifold Chow ring case, which corresponds to $V = 0$.

We also describe an *asymptotic* product on the rational Chow group of the inertia stack obtained by letting the rank of the bundle V go to infinity.

1. INTRODUCTION

In their landmark paper, Borisov, Chen and Smith [BCS] defined a *toric Deligne-Mumford (DM) stack* $\mathcal{X}(\Sigma)$ associated to a stacky fan $\Sigma = (N, \Sigma, \beta)$. Here N is a finitely generated abelian group, Σ is a simplicial fan in $N \otimes_{\mathbb{Z}} \mathbb{Q}$ with n -rays and $\beta: \mathbb{Z}^n \rightarrow N$ is a homomorphism such that the image of the standard basis of \mathbb{Z}^n generates the rays of Σ . Having defined toric DM stacks, the authors gave a beautiful presentation of the orbifold Chow ring of a toric DM stack $\mathcal{X}(\Sigma)$. The orbifold Chow ring was defined by Abramovich, Graber and Vistoli [AGV] and is an algebraic analogue of Chen-Ruan orbifold cohomology defined in [CR]. Precisely, they prove that if $\mathcal{X}(\Sigma)$ is a toric DM stack with projective coarse moduli space $X(\Sigma)$, the rational orbifold Chow ring is a quotient of the *deformed group algebra* $\mathbb{Q}[N]^{\Sigma}$ by a linear ideal. In subsequent work, Jiang and Tseng [JT] used Iwanari's calculation [Iwa] of the integral Chow ring of a toric DM stack to give a presentation for the integral orbifold Chow ring.

In this paper we turn our attention to describing general *inertial products* on toric DM stacks. The formalism of inertial products was introduced in [EJK1] and further developed in [EJK2]. In the latter paper the authors show that if \mathcal{X} is a smooth DM stack then any vector bundle V on \mathcal{X} defines two inertial products \star_{V+} and \star_{V-} on the Chow groups of the inertia stack $I\mathcal{X}$. When V is trivial the two products agree with the usual orbifold product. When $V = \mathbb{T}$ is the tangent bundle then $\star_{\mathbb{T}-}$ is the virtual product defined in [GLS⁺]. Our main result is an explicit presentation for the

Date: December 22, 2016.

Research of the second author partially supported by Simons Collaboration Grant 315460.

\star_{V+} and \star_{V-} inertial products for large class of vector bundles on a toric DM stack $\mathcal{X}(\Sigma)$.

To do this we prove the following general result about inertial Chow rings of toric DM stacks.

Theorem 1.1. *Let $\mathcal{X}(\Sigma)$ be a toric Deligne-Mumford stack associated to a stacky fan $\Sigma = (N, \Sigma, \beta)$ such that Σ spans $N_{\mathbb{R}}$ and the image of β generates N_{tors} . Then if \star is an inertial product on $\mathcal{X}(\Sigma)$, then there is an isomorphism of graded rings*

$$(1.1) \quad A^*(\mathcal{X}(\Sigma), \star, \mathbb{Z}) \cong \frac{R_{\Sigma}}{\text{CR}(\Sigma) + \text{BR}(\star, \Sigma)}.$$

where R_{Σ} is an algebra over the stacky Stanley-Reisner ring S_{Σ} whose generators correspond to the twisted sectors of $\mathcal{X}(\Sigma)$. The ring R_{Σ} and the ideal $\text{CR}(\Sigma)$ can be written explicitly in terms of the combinatorics of the stacky fan, but the box relations ideal $\text{BR}(\star, \Sigma)$ depends also on the choice of inertial product \star .

In Proposition 7.9 we give an explicit description of the ideals $\text{BR}(\star_{V-}, \Sigma)$ and $\text{BR}(\star_{V+}, \Sigma)$ when $V = \sum a_i \mathcal{L}_i$ where \mathcal{L}_i are standard line bundles on $\mathcal{X}(\Sigma)$. As a corollary we obtain an explicit presentation for the virtual product.

Taking a limit over certain inertial products allows us to define (Theorem 8.1) two new associative products on the rational Chow groups of $I\mathcal{X}$ which we call the $\star_{+\infty}$ and $\star_{-\infty}$ products.

Remark 1.2. This article is based in large part on the first author's PhD thesis [Col]

1.1. Outline of the paper. In Section 2 we begin by recalling the equivariant intersection theory of [EG1]. We then show how the equivariant projection formula can be used to compute the equivariant Chow rings of stacks of the form $[Z/G]$ where Z is the complement of a union of a linear subspaces in a representation of a diagonalizable group G (Propositions 2.2, 2.3). Since all toric stacks are naturally expressed as quotients of this form, these results play an important role in our calculations.

In Section 3 we recall the formalism of inertial products of [EJK1, EJK2]. We also recall the logarithmic trace map which is used to define the \star_{V+} and \star_{V-} products. In Section 4 we recall the construction of toric DM stacks and use the results of Section 2 to give a presentation for the ordinary Chow ring of a toric DM stack (Propositions 4.8 and 4.9) thereby giving simpler proofs of earlier results of Iwanari [Iwa] and Jiang-Tseng [JT].

In Section 5 we recall Borisov-Chen-Smith's description of the inertia of a toric DM stack $\mathcal{X}(\Sigma)$ in terms of the *box* of the stacky fan Σ . In the next section compute the logarithmic traces of the standard line bundles \mathcal{L}_i in terms of the box.

Having laid the necessary foundation we state and prove our main results in the final two sections. In Section 7 we give our presentations for inertial Chow rings and in Section 8 we describe our new asymptotic product.

2. BACKGROUND MATERIAL ON EQUIVARIANT CHOW GROUPS

Equivariant Chow groups were defined in [EG1]. We briefly recall their definition and refer the reader to [EG1] for more details. If G is a linear algebraic group acting on a scheme or, more generally, algebraic space, we denote by $A_G^i(X)$ the “codimension- i ”equivariant Chow group. It is defined as the codimension- i Chow group of the space $X \times_G U$ where U is an open set in a representation V on which G acts freely and such that $V \setminus U$ has codimension more than i . It is proved in [EG1] that this definition is independent of the choice representation V or open set U . Moreover, the equivariant Chow group $A_G^k(X)$ is an invariant of the quotient stack $[X/G]$ and we use this as our definition of Chow groups of quotient stacks.

When X is smooth, the spaces $X \times_G U$ are also smooth and we can define an intersection product on equivariant Chow groups. In this case we obtain a graded ring $A_G^*(X) = \bigoplus_{k=0}^{\infty} A_G^k(X)$ which we call the *equivariant Chow ring of X* .

By their definition, equivariant Chow groups enjoy the same formal properties as ordinary Chow groups. For example, there are pushforwards for proper equivariant morphisms and pullbacks for flat and lci morphisms, which are related by a projection formula. Equivariant vector bundles define Chern classes which are operations on the equivariant Chow groups.

An important formula for this paper is the equivariant self-intersection formula, which states that if $Y \xrightarrow{i} X$ is a regular embedding and $\alpha \in A_G^*(Y)$ then

$$i^* i_* \alpha = c_{top}(N_{Y/X}) \cap \alpha.$$

If V is an r -dimensional representation of G then V defines an equivariant vector bundle over a point and so we have corresponding Chern class $c_1(V), \dots, c_r(V) \in A_G^*(pt)$. If X is any G -space then we can pull these Chern classes back and obtain operations on $A_G^*(X)$. By abuse of notation we will also denote them as $c_i(V)$.

If V is a representation of G and $W \subset V$ is a G -invariant linear subspace then the normal bundle to W in V is the bundle $W \times V/W$. In this case the projection formula reads

$$i^* i_* \alpha = c_{top}(V/W) \cap \alpha.$$

2.1. Equivariant Chow groups for actions of diagonalizable groups. All toric stacks have presentations as $[Z/G]$ where G is a diagonalizable group, and Z is the complement of the union of a finite collection L_1, \dots, L_m of G -invariant linear subspaces

in a representation V of G . In this section state and prove some propositions which give a method for computing $A_G^*(Z)$ in terms of the representation-theoretic data.

Precisely, let $\mathbf{X} = \mathbf{X}(G)$ be the character group of G and $A_G^* = \mathbb{Z}[\mathbf{X}]$ be the algebra generated by this group. By [EG1] we can identify this with the equivariant Chow ring $A_G^*(pt)$.

Proposition 2.1. *If each L_i has codimension k_i then*

$$A_G^*(Z) = A_G^* / (c_{k_1}(V/L_1), \dots, c_{k_m}(V/L_m)).$$

Proof. Let L be single linear subspace. Since L and V are G -vector bundles over a point, the homotopy property of equivariant Chow groups implies that the pullbacks $A_G^* \rightarrow A_G^*L$ and $A_G^* \rightarrow A_G^*V$ are isomorphisms. Hence, if $i: L \hookrightarrow V$ is the inclusion then i^* is also an isomorphism. Let $j: V \setminus L \rightarrow V$ be the open inclusion. The excision short exact sequence yields a surjection of equivariant Chow rings $A_G^*(V) = A_G^* \twoheadrightarrow A_G^*(Z)$ whose kernel is $i_*A_G^*(L)$. Since i^* is an isomorphism, the projection formula implies that $i_*A_G^*(L) = i_*([L])$ where $[L]$ is the fundamental class. By the self-intersection formula, $i^*i_*([L]) = c_{top}(N_LV)$. Since the normal bundle to L in V is just the representation V/L we see that $i^*i_*([L]) = c_{top}(V/L)$, so $A_G^*(Z) = A_G^* / (c_{top}(V/L))$.

The general case follows by induction. ■

When the representation V of G is faithful then we can identify G as a closed subgroup of \mathbb{G}_m^n where $n = \dim V$. The inclusion $G \hookrightarrow \mathbb{G}_m^n$ induces a surjection of character groups $\mathbb{Z}^n \rightarrow \mathbf{X}$. If we let x_i be the first Chern class of the image in \mathbf{X} of the standard basis vector e_i then A_G^* is generated by x_1, \dots, x_n and $A_G^* = \mathbb{Z}[x_1, \dots, x_n]/C(G)$ where $C(G)$ is the linear ideal representing the relations between the images of the e_i in the finitely generated abelian group \mathbf{X} . Choose coordinates z_1, \dots, z_n such that the action of \mathbb{G}_m^n (and thus G) is diagonalized with respect to them. With this notation, a G -invariant linear subspace L has ideal $(z_{i_1}, \dots, z_{i_k})$ for some subset $\{i_1, \dots, i_k\}$ of $\{1, \dots, n\}$. The normal bundle to the hyperplane $z_i = 0$ is $x_i = c_1(e_i)$ so $c_{top}(V/L) = x_{i_1} \dots x_{i_k}$ and we can restate Proposition 2.1 as.

Proposition 2.2. *Let V is a faithful representation of a diagonalizable group G . Let $Z = V \setminus \bigcup (L_1 \cup \dots \cup L_m)$ where L_k is the linear subspace $V(z_{k_1}, \dots, z_{k_{r_k}})$. Then*

$$A_G^*(Z) \simeq \mathbb{Z}[x_1, \dots, x_n] / (C(G) + (\{x_{k_1} \dots x_{k_{r_k}}\}_{k=1, \dots, m}))$$

Finally, we consider the situation where the action of G is not faithful but the morphism $G \rightarrow \mathbb{G}_m^n$ factors as $G \xrightarrow{\iota} G \hookrightarrow \mathbb{G}_m^n$ where ι is an isogeny with finite kernel and cokernel and the second map is an immersion. In this case let \tilde{x}_i be the first Chern class of the image of e_i under the composition of maps of character groups $\mathbf{X}(\mathbb{G}_m^n) \rightarrow \mathbf{X} \rightarrow \mathbf{X}$. We obtain the following description of the equivariant Chow ring in this case.

Proposition 2.3. *With the notation as in the previous paragraph*

$$A_G^*(Z) = \mathbb{Z}[x_1, \dots, x_n] / (C(G) + (\{\tilde{x}_{k_1} \dots \tilde{x}_{k_{r_k}}\}_{k=1, \dots, m}))$$

Now suppose that $V' \subset V$ be a subrepresentation of G and let $Z' = Z \cap V'$. Pullback on the closed embedding $Z' \xrightarrow{j} Z$ makes $A_G^*(Z')$ into an $A_G^*(Z)$ -module. For each linear subspace L_i let E_i denote the quotient of normal bundles $(V/L_i)/(V'/V' \cap L_i)$.

Proposition 2.4. *The pullback $j^*: A_G^*(Z) \rightarrow A_G^*(Z')$ is surjective and $\ker j^*$ is the ideal generated by $\{c_{\text{top}}(E_i)\}_{i=1}^m$.*

Proof. This follows the descriptions of $A_G^*(V)$ and $A_G^*(V')$ given by Proposition 2.1. ■

We can combine Proposition 2.4 with Propositions 2.2 and 2.3 to obtain the an explicit description of $A_G^*(V')$ as a quotient of $A_G^*(V)$. To do so we need to introduce some notation. As above, fix coordinate z_1, \dots, z_n on V such that the action of G is diagonalized with respect to the them. Given a linear subspace L , let $M(L) = \{i | z_i|_L = 0\}$. (In other words, $I(L) = (\{z_i\}_{i \in M(L)})$.) Then $N_L V = \bigoplus_{z_i \in M(L)} \mathcal{O}(z_i)$ and we have the following proposition.

Proposition 2.5. *If G acts faithfully on V then*

$$A_G^*(V') = A_G^*(V) / (\{ \prod_{i_k \in M(L_k) \setminus M(V')} x_{i_k} \}_{k=1, \dots, m})$$

and if G acts as the composition of an isogney with a faithful action then

$$A_G^*(V') = A_G^*(V) / (\{ \prod_{i_k \in M(L_k) \setminus M(V')} \tilde{x}_{i_k} \}_{k=1, \dots, m})$$

3. INERTIAL PRODUCTS

Let G be an algebraic group acting on a scheme X . We recall the formalism of inertial products of [EJK2]. However, since our application is to toric stacks, we assume that G is diagonalizable throughout.

3.1. Inertia spaces.

Definition 3.1. The *inertia space* for the action of G on X is defined as

$$I_G X = \{(g, x) | gx = x\} \subset G \times X,$$

and the l -th *higher inertia space* is the l -tuple fiber product over X

$$I_G^l X = \{(g_1, \dots, g_l, x) | g_i x = x \text{ for } i = 1, \dots, l\} \subset G^l \times X.$$

Definition 3.2. If \mathcal{X} is defined to be the quotient stack $[X/G]$, then the *inertia stack* of \mathcal{X} is

$$I\mathcal{X} = [I_G X/G],$$

and the l -th *higher inertia stack* is

$$\mathbb{I}^l \mathcal{X} = [\mathbb{I}_G^l X/G].$$

Definition 3.3. Let X be a variety with the action of a group G . For any l -tuple $\mathbf{g} = (g_1, \dots, g_l) \in G^l$, define the twisted sector

$$X^{\mathbf{g}} = \{(g_1, \dots, g_l, x) \in \mathbb{I}_G^l X\}.$$

Note that in the case $\mathbf{g} = \text{id} \in G$, we do not consider the sector $X^{\mathbf{g}} = X$ to be twisted.

Remark 3.4. For convenience purposes, we will occasionally identify $\mathbb{I}_G^l X$ with the open and closed subset

$$\{(g_1, \dots, g_{l+1}, x) \mid g_1 g_2 \cdots g_{l+1} = 1\}$$

of $\mathbb{I}_G^{l+1} X$.

Definition 3.5. Take any $(g_1, g_2, g_3, x) \in \mathbb{I}_G^2 X$ where, as in Remark 3.4, $g_1 g_2 g_3 = 1$. Then for $i = 1, 2, 3$, we define $e_i : \mathbb{I}_G^2 X \rightarrow I_G X$ be the evaluation morphism $(g_1, g_2, g_3, x) \mapsto (g_i, x)$. Additionally, we define $\mu : \mathbb{I}_G^2 X \rightarrow I_G X$ be the morphism $(g_1, g_2, g_3, x) \mapsto (g_1 g_2, x)$.

Definition 3.6. Given a class $c \in A_G^*(\mathbb{I}^2 X)$, we define the *inertial product with respect to c* to be

$$x \star_c y := \mu_*(e_1^* x \cdot e_2^* y \cdot c),$$

where $x, y \in A_G^*(I_G X)$.

We say that a G -equivariant vector bundle \mathcal{R} on $\mathbb{I}^2 X$ is associative if the inertial product with respect to $c = \text{eu}(\mathcal{R})$ is associative. The basic example of an associative vector bundle is the obstruction bundle used to define the orbifold product. This, and a plethora of other associative bundles can be constructed using the logarithmic trace construction of [EJK1, EJK2].

3.2. Logarithmic trace and inertial products. Let V be a rank- n vector bundle on X which is G -equivariant. Let g be an element of a group $G \subseteq (\mathbb{C}^\times)^n$ which acts trivially on X and whose action is an automorphism of the fibers of $V \rightarrow X$. Under these conditions, the eigenbundles for the action of g will be G -subbundles of E .

Denote the eigenvalues for the action of g on V by $\exp(2\pi\sqrt{-1}\lambda_k)$ for $1 \leq k \leq n$, and without loss of generality we can say $0 \leq \lambda_k < 1$ for each k . We will use E_k to denote the eigenbundle corresponding to λ_k .

Definition 3.7. [EJK1, Definition 4.1] The *logarithmic trace* of V for the action of g is

$$L(g)(V) = \sum_{k=1}^r \lambda_k V_k \in K_G(X) \otimes \mathbb{R}$$

Remark 3.8. The assumption that G is diagonalizable implies that the full group G acts on X^g for any $g \in G$. This simplifies Definitions 3.7 and 3.10 as compared to those found in [EJK1, EJK2].

Example 3.9. Let $Z = \mathbb{C}^4 - \{(0, 0, 0, 0)\}$, and let $G = \mathbb{C}^\times$ act on Z , via

$$g \cdot (z_1, z_2, z_3, z_4) = (g^2 z_1, g^4 z_2, g^5 z_3, g^6 z_4).$$

Let V be a vector bundle on Z with subbundles L_i corresponding to the divisors where $z_i = 0$. That is, V is the tangent bundle $T_V = L_1 + L_2 + L_3 + L_4$.

First, note that $I_G Z = \coprod Z^g$, where the union is over the twelve elements $g \in G = \mathbb{C}^\times$ which fix at least one point of Z . In order to fix a point of Z , g must be either a second, fourth, fifth or sixth root of unity.

Let's look at $g = e^{2\pi\sqrt{-1}/3} \in G$, and compute its logarithmic trace. Note that it only makes sense to consider the action of g on V if we restrict to $V|_{Z^g}$.

Under the induced action of α on Z , we have

$$g \cdot (z_1, z_2, z_3, z_4) = (e^{4\pi\sqrt{-1}/3} z_1, e^{2\pi\sqrt{-1}/3} z_2, e^{4\pi\sqrt{-1}/3} z_3, z_4).$$

So g has three eigenvalues and eigenbundles for its action on $V|_{Z^g}$:

$$\begin{aligned} \lambda_1 &= 1 && \text{for } E_1 = L_4 \\ \lambda_2 &= e^{4\pi\sqrt{-1}/3} && \text{for } E_2 = L_1 \text{ or } E_2 = L_3 \\ \lambda_3 &= e^{2\pi\sqrt{-1}/3} && \text{for } E_3 = L_2 \end{aligned}$$

Thus, the logarithmic trace of g on $V|_{Z^g}$ is

$$L(g)(V|_{Z^g}) = \frac{2}{3}L_1 + \frac{1}{3}L_2 + \frac{2}{3}L_3.$$

Notice that the coefficient on L_4 in the logarithmic trace is 0; this is a direct consequence of the fact that $Z^g = \{(0, 0, 0, z_4) \mid z_4 \neq 0\}$.

Definition 3.10. [EJK1, Definition 5.3] Let \mathbf{g} be an l -tuple (g_1, \dots, g_l) such that there exists a finite subgroup of G containing every g_i for $1 \leq i \leq l$, and with the additional stipulation that $g_1 \cdots g_l = 1$. Then the *logarithmic restriction* of E is the class in $K_G(X^{\mathbf{g}})$ defined by the formula

$$(3.1) \quad V(\mathbf{g}) = \sum_{i=1}^l L(g_i)(V|_{X^{\mathbf{g}}}) + V^{\mathbf{g}} - V|_{X^{\mathbf{g}}}.$$

The assignment $LR : V \mapsto V(\mathbf{g})$ is called the *logarithmic restriction* map, and takes non-negative elements $E \in K_G(X)$ to non-negative elements $LR(V) \in K_G(X^{\mathbf{g}})$.

Example 3.11. We'll build on Example 3.9, with the same Z , G , α and V ; additionally, let $\mathbf{g} = (e^{\pi\sqrt{-1}/3}, e^{2\pi\sqrt{-1}/3}, e^{\pi\sqrt{-1}})$. Then we have $Z^{\mathbf{g}} = \{(0, 0, 0, z_4) \mid z_4 \neq 0\} \subseteq Z$, which in turn gives $V^{\mathbf{g}} = L_4$.

Then the logarithmic restriction of V at \mathbf{g} is

$$\begin{aligned} V(\mathbf{g}) &= L\left(e^{\pi\sqrt{-1}/3}\right)(V|_{Z^{\mathbf{g}}}) + L\left(e^{2\pi\sqrt{-1}/3}\right)(V|_{Z^{\mathbf{g}}}) + L\left(e^{\pi\sqrt{-1}}\right)(V|_{Z^{\mathbf{g}}}) + V^{\mathbf{g}} - V|_{Z^{\mathbf{g}}} \\ &= \left(\frac{1}{3}L_1 + \frac{2}{3}L_2 + \frac{5}{6}L_3\right) + \left(\frac{2}{3}L_1 + \frac{1}{3}L_2 + \frac{2}{3}L_3\right) + \frac{1}{2}L_3 \\ &\quad + L_4 - (L_1 + L_2 + L_3 + L_4) \\ &= L_3. \end{aligned}$$

Proposition 3.12. [EJK2, Proposition 4.0.10] *Let V be a G -equivariant bundle on X , and let g_1, g_2 lie in a common compact subgroup of G . Then the virtual bundles*

$$(3.2) \quad V^+(g_1, g_2) = L(g_1)(V|_{X^{g_1, g_2}}) + L(g_2)(V|_{X^{g_1, g_2}}) - L(g_1 g_2)(V|_{X^{g_1, g_2}})$$

and

$$(3.3) \quad V^-(g_1, g_2) = L(g_1^{-1})(V|_{X^{g_1, g_2}}) + L(g_2^{-1})(V|_{X^{g_1, g_2}}) - L(g_1^{-1} g_2^{-1})(V|_{X^{g_1, g_2}})$$

are represented by non-negative integral elements in $K_G(X^{g_1, g_2})$.

Definition 3.13. Let $\mathbf{g} = (g_1, g_2) \in G^2$. Define classes R^+V and R^-V in $K_G(\mathbb{I}^2 X)$ by setting the component of R^+V (resp. R^-V) in $K_G((X^{\mathbf{g}}))$ to be $V^+(g_1, g_2)$ (resp. $V^-(g_1, g_2)$).

Theorem 3.14. [EJK2, Theorem 4.0.12] *Let \mathbb{T} be the G -equivariant vector bundle corresponding to the tangent bundle of $[X/G]$ and let \mathcal{R}^+V be the vector bundle $LR(\mathbb{T}) + R^+V$ for any G -equivariant vector bundle V on X . Then the inertial product with respect to $\text{eu}(\mathcal{R}^+V)$, which we call \star_{V^+} , is associative. Similarly, $\text{eu}(\mathcal{R}^-V) = \text{eu}(LR(\mathbb{T}) + R^-V)$, called \star_{V^-} , also defines an associative inertial product.*

Remark 3.15. [EJK2] The orbifold product \star_{orb} is defined by taking $V = 0$ in Definition 3.6 and Theorem 3.14 (with either \mathcal{R}^+ or \mathcal{R}^-). The virtual product \star_{virt} of [GLS⁺] is the $\star_{\mathbb{T}^-}$ product.

4. STACKY FANS AND TORIC STACKS

4.1. Definitions and setup. We recall the construction of toric DM stacks from stacky fans following [BCS].

A stacky fan Σ is a triple (N, Σ, β) , where N is a finitely generated abelian group of rank d , Σ is a simplicial fan with n rays, ρ_1, \dots, ρ_n in the d -dimensional \mathbb{Q} -vector space $N_{\mathbb{Q}} := N \otimes_{\mathbb{Z}} \mathbb{Q}$, and β is a homomorphism from \mathbb{Z}^n to N defined by a choice of elements $b_1, \dots, b_n \in N$ such that the image of b_i in $N \otimes \mathbb{Q}$ generates the ray ρ_i . We call b_1, \dots, b_n the *distinguished points* of Σ in N .

Let X_{Σ} be the toric variety associated to the fan Σ . Let $T_N := \text{Hom}(N^*, \mathbb{C}^{\times})$ and $T_L := \text{Hom}((\mathbb{Z}^n)^*, \mathbb{C}^{\times}) \cong (\mathbb{C}^{\times})^n$. Then the natural map $\beta^* : N^* \rightarrow (\mathbb{Z}^n)^*$ induces a morphism of tori $T_{\beta} : T_L \rightarrow T_N$.

Let $N = \mathbb{Z}^d \oplus \mathbb{Z}/m_1 \oplus \dots \oplus \mathbb{Z}/m_r$ be the invariant factor form of N . We define $\beta_{aug} : \mathbb{Z}^n \oplus \mathbb{Z}^r \rightarrow \mathbb{Z}^{d+r}$ to be the map represented by the matrix

$$\begin{bmatrix} b_{1,1} & & b_{n,1} & 0 & 0 & 0 \\ & \ddots & & 0 & \ddots & 0 \\ b_{1,d} & & b_{n,d} & 0 & 0 & 0 \\ b_{1,d+1} & & b_{n,d+1} & m_1 & 0 & 0 \\ & \ddots & & 0 & \ddots & 0 \\ b_{1,d+r} & & b_{n,d+r} & 0 & 0 & m_r \end{bmatrix}$$

where the i -th column of β_{aug} is a lift of b_i under the natural surjection $\mathbb{Z}^d \oplus \mathbb{Z}^r \rightarrow N$.

For convenience, we will represent the integers $b_{i,d+j}$ (which are elements of \mathbb{Z}/m_l) using elements of the set $\{0, 1, \dots, m_l - 1\}$ for $l = 1, \dots, r$, but the construction is independent of choice (cf Remark 4.3).

Define a map $E^{\beta} : (\mathbb{C}^{\times})^{n+r} \rightarrow (\mathbb{C}^{\times})^{d+r}$ by exponentiating β_{aug} . That is,

$$E^{\beta}(\gamma_1, \dots, \gamma_n, s_1, \dots, s_r) = \left(\prod_{i=1}^n \gamma_i^{b_{i,1}}, \dots, \prod_{i=1}^n \gamma_i^{b_{i,d}}, s_1^{m_1} \prod_{i=1}^n \gamma_i^{b_{i,d+1}}, \dots, s_r^{m_r} \prod_{i=1}^n \gamma_i^{b_{i,d+r}} \right).$$

Let G be the kernel of the map E^{β} . As a subgroup of $(\mathbb{C}^{\times})^{n+r}$:

$$(4.1) \quad G = \left\{ (\gamma_1, \dots, \gamma_n, s_1, \dots, s_r) \left| \begin{array}{l} \prod_{i=1}^n \gamma_i^{b_{i,j}} = 1 \text{ for } 1 \leq j \leq d, \text{ and} \\ s_l^{m_l} \cdot \prod_{i=1}^n \gamma_i^{b_{i,d+l}} = 1 \text{ for } 1 \leq l \leq r \end{array} \right. \right\}.$$

Let $\mathbb{C}[z_{\rho} \mid \rho \in \Sigma(1)] = \mathbb{C}[z_1, \dots, z_n]$ be the total coordinate ring of X_{Σ} . There is a natural action of G on $\mathbb{A}^n = \text{Spec } \mathbb{C}[z_1, \dots, z_n]$ given by

$$(\gamma_1, \dots, \gamma_n, s_1, \dots, s_r) \cdot (z_1, \dots, z_n) = (\gamma_1 z_1, \dots, \gamma_n z_n).$$

Let $Z = \mathbb{A}^n \setminus V(J_\Sigma)$ where $J_\Sigma := \left\langle \prod_{\rho_i \not\leq \sigma} z_i \mid \sigma \in \Sigma \right\rangle$ is the irrelevant ideal. Since $V(J_\Sigma)$ is a union of coordinate subspaces, $Z = \mathbb{A}^n \setminus V(J_\Sigma)$ is G -invariant so there is an action of G on Z .

The *toric stack associated to Σ* is defined to be the stack quotient $\mathcal{X}(\Sigma) := [X_\Sigma/G] \cong [Z/G]$ under this action. With the assumptions of our setup $\mathcal{X}(\Sigma)$ will be a smooth, separated Deligne-Mumford stack [BCS, Proposition 3.2].

Remark 4.1. If N is torsion-free, we can simplify this explicit construction. Since $r = 0$, we get

$$(4.2) \quad G = \left\{ (\gamma_1, \dots, \gamma_n) \mid \prod_{i=1}^n \gamma_i^{b_{i,j}} = 1 \text{ for } 1 \leq j \leq d \right\}$$

In this case the action of G on \mathbb{A}^n corresponds to a faithful representation. Thus, the stabilizer of a general point is trivial and the quotient stack $[Z/G]$ is an effective orbifold.

If N has torsion then $\mathcal{X}(\Sigma)$ then the action of G on Z is not generically free as the following example shows.

Example 4.2. Let $N = \mathbb{Z} \oplus \mathbb{Z}/2$ and let Σ be the complete fan in $N_\mathbb{Q} \cong \mathbb{Q}$. Define β by the distinguished points $b_1 = (2, 1)$ and $b_2 = (-3, 0)$ in N . This defines a stacky fan $\Sigma = (N, \Sigma, \beta)$. Notice that b_1 and b_2 are *not* minimal generators of ρ_1 and ρ_2 over $N_\mathbb{Q}$.

As N is not a lattice, we must compute G using (4.1). We have the homomorphism β_{aug} :

$$\begin{bmatrix} 2 & -3 & 0 \\ 1 & 0 & 2 \end{bmatrix} : \mathbb{Z}^3 \rightarrow \mathbb{Z}^2$$

Hence,

$$G = \ker(E^\beta) = \{(\gamma_1, \gamma_2, s_1) \in (\mathbb{C}^\times)^3 \mid \gamma_1^2 \gamma_2^{-3} = 1, \gamma_1 s_1^2 = 1\}.$$

Computing J_Σ the usual way, we have $Z = \mathbb{C}^2 \setminus \{(0, 0)\}$, so the G -action on Z is

$$(\gamma_1, \gamma_2, s_1) \cdot (z_1, z_2) = (\gamma_1 z_1, \gamma_2 z_2).$$

But since $G \cong \{(\lambda^6, \lambda^4, \lambda^{-3}) \mid \lambda \in \mathbb{C}^\times\} \cong \mathbb{C}^\times$, the G -action is equivalent to the \mathbb{C}^\times -action

$$\lambda \cdot (z_1, z_2) = (\lambda^6 z_1, \lambda^4 z_2).$$

That is, $\mathcal{X}(\Sigma)$ is the weighted projective stack $\mathbb{P}(6, 4)$, which of course is not a reduced orbifold.

Remark 4.3. In Example 4.2, we made a specific choice of distinguished points $b_1 = (2, 1)$ and $b_2 = (-3, 0)$. If we had chosen any other equivalent lift under the surjection

$\mathbb{Z} \oplus \mathbb{Z} \rightarrow \mathbb{Z} \oplus \mathbb{Z}/2$, we would have obtained the same stack. As an extreme example, we can pick $b_1 = (2, 87)$ and $b_2 = (-3, 54)$ which gives us

$$G \cong \{(\lambda^6, \lambda^4, \lambda^{-369}) | \lambda \in \mathbb{C}^\times\},$$

which yields the same stack $\mathcal{X}(\Sigma) = \mathbb{P}(6, 4)$.

4.2. Characters and Line bundles. Let $\mathcal{X}(\Sigma) = [Z/G]$ be a toric Deligne-Mumford stack associated to the stacky fan $\Sigma = (N, \Sigma, \beta)$ and let $X(\Sigma)$ be the underlying toric variety.

A line bundle on $[Z/G]$ is a G -equivariant line bundle on Z . Since Z is an open set in a representation of G , $\text{Pic}([Z/G]) = A_G^1(Z)$ is generated by characters of G . In, addition if we assume that the fan Σ has maximal dimension (for example if $X(\Sigma)$ is proper) then the complement of Z in $\mathbb{A}^{|\Sigma(1)|}$ has codimension at least two, so $\text{Pic}([Z/G]) = \text{Pic}^G(\mathbb{A}^{|\Sigma(1)|}) = \mathbf{X}(G) = N^\vee$. When $[Z/G]$ is a reduced orbifold, meaning that the generic stabilizer is trivial (ie the action of G is effective) then $\text{Pic}(\mathcal{X}(\Sigma)) = \text{Pic}(X(\Sigma))$. In this case if we let L_i be the line bundle on $X(\Sigma)$ corresponding to the ray ρ_i then $\text{Pic}(\mathcal{X}(\Sigma))$ is generated by the pullback of these line bundles to $\mathcal{X}(\Sigma)$ which we also denote by L_i . If we identify $\mathbf{X}(\mathbb{G}_m^n) = \mathbb{Z}^n$ and let e_i be character corresponds to the i -th standard basis vector, then L_i corresponds to the image of e_i in $\mathbf{X}(G)$ under the surjective morphism $\mathbb{Z}^n \rightarrow \mathbf{X}(G) = N^\vee$.

In general the action of G on Z is not effective. However, if we assume that b_1, \dots, b_n generate N_{tors} then Jiang and Tseng [JT, Section 3] prove that the action of G on \mathbb{A}^n corresponds to a map $G \xrightarrow{\iota} G \xrightarrow{i} \mathbb{G}_m^n$ where ι is an isogeny and i is an immersion. The quotient $[Z/_i G]$ where the subscript $_i$ indicate that the action is the faithful action of G produces the *rigidification* of $\mathcal{X}(\Sigma)_{\text{red}}$ of $\mathcal{X}(\Sigma)$ in the sense of [AGV]. Let \mathcal{L}_i be the image of e_i under the composite map $\mathbb{Z}^n \rightarrow \mathbf{X}(G) \xrightarrow{\iota^*} \mathbf{X}(G)$. Equivalently, \mathcal{L}_i is the pullback of the line bundle corresponding to the ray ρ_i in the underlying toric variety. If z_1, \dots, z_n are coordinates on \mathbb{A}^n then $\mathcal{O}(z_i) = \mathcal{L}_i$.

Lemma 4.4. [JT, Section 3.3] *If β_1, \dots, β_n generate N_{tors} then $\text{Pic}(\mathcal{X}(\Sigma))$ is generated by the line bundles L_1, \dots, L_n .*

Remark 4.5. Let $x_i = c_1(L_i)$ and $\tilde{x}_i = c_1(\mathcal{L}_i)$, then

$$\tilde{x}_i = \sum_{k=1}^n f_{k,i} x_k$$

for some integers $f_{k,i}$ such that $\det(f_{k,j}) \neq 0$. Jiang and Tseng refer to this equation as the *associated formula* for the \tilde{x}_i .

Example 4.6. The stack $\mathcal{X}(\Sigma) = \mathbb{P}(6, 4)$ of Example 4.2 has distinguished points $b_1 = (2, 1)$ and $b_2 = (-3, 0)$ in $N = \mathbb{Z} \oplus \mathbb{Z}/2$. The underlying toric variety is $\mathbb{P}(3, 2)$, with line bundles L_1 and L_2 having Chern classes $x_1 := c_1(L_1) = 3t$ and $x_2 := c_1(L_2) = 2t$

where t is the first Chern class of the defining character of $G = \mathbb{C}^\times$. The associated formula of the stacky fan Σ is

$$\tilde{x}_1 = 2x_1 = 6t \quad \tilde{x}_2 = 2x_2 = 4t$$

Example 4.7. Now let $N = \mathbb{Z} \oplus \mathbb{Z}/2$ and let Σ be the complete fan in $N_{\mathbb{Q}} \cong \mathbb{Q}$ but define β by the distinguished points $b_1 = (2, 0)$ and $b_2 = (-3, 0)$ in N so the image of β_1, β_2 does not generate N_{tors} . In this case our augmentation homomorphism is

$$\begin{bmatrix} 2 & -3 & 0 \\ 0 & 0 & 2 \end{bmatrix} : \mathbb{Z}^3 \rightarrow \mathbb{Z}^2$$

Hence,

$$G = \ker(E^\beta) = \{(\gamma_1, \gamma_2, s_1) \in (\mathbb{C}^\times)^3 \mid \gamma_1^2 \gamma_2^{-3} = 1, s_1^2 = 1\}.$$

so $G = \mathbb{C}^\times \times \mu_2$ with G acting by $(\lambda, \mu)(z_1, z_2) = (\lambda^2 z_1, \lambda^3 z_2)$. Thus $\mathcal{X}(\Sigma) = \mathbb{P}(2, 3) \times B\mu_2$ and $\text{Pic}(\mathcal{X}(\Sigma)) = \mathbb{Z} \oplus \mathbb{Z}_2$. Here we have $L_i = \mathcal{L}_i$ but L_1, L_2 do not generate the full Picard group because they do not detect the torsion.

4.3. The Chow ring of a toric stack. Here, we use Propositions 2.2 and 2.3 to give simple proofs of the results of [Iwa, JT] on the integral Chow rings of toric stacks. Note that, unlike [JT] we do not require that the stack be proper but we require that every maximal cone of Σ has dimension $d = \text{rank } N$. In particular this implies that the vectors b_1, \dots, b_n span $N \otimes \mathbb{R}$.

Proposition 4.8. [Iwa, Theorem 2.2] *If N is torsion free and b_1, \dots, b_n span $N \otimes \mathbb{R}$ then $A^*(\mathcal{X}(\Sigma)) = \mathbb{Z}[x_1, \dots, x_n] / (I_\Sigma + C(\Sigma))$.*

where I_Σ is the ideal generated by monomials

$$\{x_{i_1}, \dots, x_{i_k} : \rho_{i_1}, \dots, \rho_{i_k} \text{ do not lie in a cone of } \Sigma\}.$$

and $C(\Sigma)$ is the ideal generated by the linear relations

$$\left(\sum_{i=1}^n \theta(\bar{b}_i) x_i \right)_{\theta \in N^*}$$

Proof. We use Proposition 2.2.

The definition of G implies that $\mathbf{X}(G)$ is cokernel of the map $N^* = (\mathbb{Z}^d)^* \xrightarrow{B^*} (\mathbb{Z}^n)^*$. Thus $\mathbf{X}(G)$ is the quotient of \mathbb{Z}^n by the linear relations $\{\sum_i \theta(b_i) x_i\}_{\theta \in N^*}$, so $C(\Sigma) = C(G)$ where $C(G)$ is as in Proposition 2.2.

If we use coordinates z_1, \dots, z_n on \mathbb{A}^n then $V(J_\Sigma)$ decomposes as the union of the linear subspaces $\{L_\sigma\} = V(z_{k_1}, \dots, z_{k_r})$ where $\rho_{k_1}, \dots, \rho_{k_r}$ are the rays of $\Sigma(1) \setminus \sigma(1)$ and σ runs through all maximal cones. Thus, I_Σ is exactly the ideal generated by the products $x_{k_1} x_{k_2} \dots x_{k_r}$ as in Proposition 2.2.

■

Proposition 4.9. [JT, Theorem 1.1] *Now let N be an arbitrary finitely generated abelian group and assume that b_1, \dots, b_n generate N_{tors} . Then*

$$A^*(\mathcal{X}(\Sigma)) = \mathbb{Z}[x_1, \dots, x_n] / (I_\Sigma + C(\Sigma)).$$

where I_Σ is the ideal generated by monomials

$$\{\tilde{x}_{i_1}, \dots, \tilde{x}_{i_k} : \rho_{i_1}, \dots, \rho_{i_k} \text{ do not lie in a cone of } \Sigma\}.$$

where \tilde{x}_i is the first Chern class of the image of the character e_i in $\mathbf{X}(G)$ and $C(\Sigma)$ is the ideal generated by the linear relations

$$\left(\sum_{i=1}^n \theta(\bar{b}_i) x_i \right)_{\theta \in N^*}$$

Proof. The key point, proved by Jiang and Tseng, is that the assumption that b_1, \dots, b_n means that the representation $G \rightarrow \mathbb{G}_m^n$ factors as an isogeny $G \rightarrow G$ followed by faithful representation $G \hookrightarrow \mathbb{G}_m^n$ corresponding to the map $\mathbb{Z}^n \rightarrow \bar{N} = N/N_{tors}$. The proposition now follows from Proposition 2.3. ■

5. THE INTERIA OF A TORIC STACK

5.1. The box of a stacky fan. Let $\Sigma = (N, \Sigma, \beta)$ be a stacky fan such that Σ spans $N \otimes \mathbb{Q}$. Following [BCS] we introduce $\text{Box}(\Sigma)$, which plays an important role in the presentations for our inertial Chow rings.

Let $\bar{N} := N/N_{tor}$ and for any $v \in N$, we use \bar{v} to denote the image of v in the natural projection $N \rightarrow \bar{N}$. In a slight abuse of notation, we will sometimes refer to \bar{v} as being in Σ .

Denote by $\sigma(\bar{v})$ the unique minimal cone $\sigma \in \Sigma$ which contains \bar{v} . More generally, $\sigma(\bar{v}_1, \dots, \bar{v}_k)$ is the minimal cone in Σ containing each of $\bar{v}_1, \dots, \bar{v}_k$.

Definition 5.1. For any cone $\sigma \in \Sigma$, we define $\text{Box}(\sigma)$ to be the set of all elements $v \in N$ such that $\bar{v} = \sum_{i|\rho_i \in \sigma(\bar{v})(1)} q_i \bar{b}_i$ with $q_i \in [0, 1) \cap \mathbb{Q}$ (Here we use $\sigma(\bar{v})(1)$ to denote the rays of the cone $\sigma(\bar{v})$.)

Define $\text{Box}(\Sigma)$ to be the union of $\text{Box}(\sigma)$ for all cones $\sigma \in \Sigma$.

For any cone $\sigma \in \Sigma$, define

$$(5.1) \quad N_\sigma := \langle b_i \mid \rho_i \in \sigma \rangle$$

as a subgroup of N , and define $N(\sigma)$ be the finite quotient group N/N_σ . The set $\text{Box}(\sigma)$ has a natural bijection with $N(\sigma)$, and gives a choice of coset representatives for the quotient group.

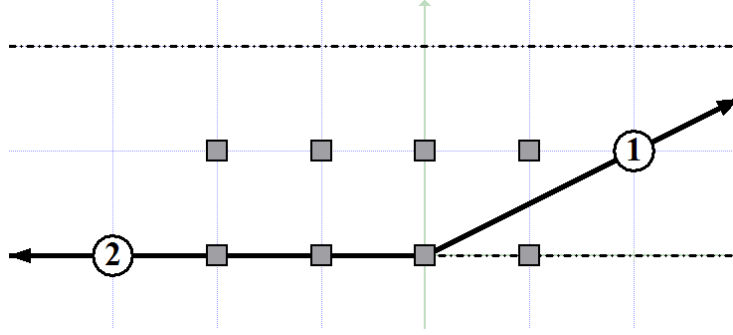


FIGURE 1. The stacky fan of Example 4.2, which yields the toric stack $\mathbb{P}(6, 4)$. The vertical axis here is representing $\mathbb{Z}/2$; that is to say, the two dotted lines are identified. The squares are elements of $\text{Box}(\Sigma)$.

Example 5.2. Let $\mathcal{X}(\Sigma)$ be the toric stack $\mathbb{P}(6, 4)$, constructed in Example 4.2. The squares in Figure 1 represent the eight elements of $\text{Box}(\Sigma)$. For example, since $\overline{N} = \mathbb{Z}$, we have that $v = (1, 1) \in N$ is in $\text{Box}(\Sigma)$ because $\bar{v} = \frac{1}{2}\bar{b}_1 \in \text{Box}(\sigma_1)$, where σ_1 is the one-dimensional cone in the positive direction on the horizontal axis.

Notice also that $(5, 1) \in N$, but $(5, 1) \notin \text{Box}(\Sigma)$. The box element which is equivalent to $(5, 1)$ in $N(\sigma_1)$ is $(1, 1)$, since $(5, 1) = 2b_1 + (1, 1) \in N$.

The set $\text{Box}(\sigma)$ inherits the abelian group structure of $N(\sigma)$. We give an example to illustrate how the addition works.

Example 5.3. We'll revisit the stacky fan of Example 4.2. We have $\text{Box}(\sigma_1) = \{0, v_1, v_2, v_3\}$, where $0 = (0, 0)$, $v_1 = (1, 1)$, $v_2 = (1, 0)$ and $v_3 = (0, 1)$. Then we have the following addition table for $\text{Box}(\sigma_1)$:

$$(5.2) \quad \begin{array}{c|cccc} + & 0 & v_1 & v_2 & v_3 \\ \hline 0 & 0 & v_1 & v_2 & v_3 \\ v_1 & v_1 & v_3 & 0 & v_2 \\ v_2 & v_2 & 0 & v_3 & v_1 \\ v_3 & v_3 & v_2 & v_1 & 0 \end{array}$$

Proposition 5.4. [BCS, Lemma 4.6] *If Σ is a complete fan, then the elements $v \in \text{Box}(\Sigma)$ are in one-to-one correspondence with elements $g \in G$ which fix a point of Z .*

Proof. Let G_σ be the subgroup of G defined by the equations $\gamma_i = 1$ for all $\rho_i \notin \sigma$. That is,

$$(5.3) \quad G_\sigma = \left\{ (\gamma_1, \dots, \gamma_n, s_1, \dots, s_r) \left| \begin{array}{l} \prod_{\rho_i \in \sigma} \gamma_i^{b_{i,j}} = 1 \text{ for } 1 \leq j \leq d, \text{ and} \\ s_l^{m_l} \cdot \prod_{\rho_i \in \sigma} \gamma_i^{b_{i,d+l}} = 1 \text{ for } 1 \leq l \leq r, \text{ and} \\ \gamma_i = 1 \text{ for all } \rho_i \notin \sigma \end{array} \right. \right\}.$$

The set of elements of G which fix a point of Z will be $\bigcup_{\sigma \in \Sigma} G_\sigma$.

We'll prove the statement for $g \in G_\sigma$ and $v \in \text{Box}(\sigma)$; taking unions will complete the proof.

For $1 \leq j \leq d$, recall that $b_{i,j}$ denotes the j -th entry of the distinguished point b_i corresponding to the ray ρ_i . We claim that for $g = (\gamma_1, \dots, \gamma_n, s_1, \dots, s_r) \in G_\sigma$, the correspondence is $g \leftrightarrow v$, where v is given by

$$(5.4) \quad \left(\sum_{i=1}^n b_{i,1} \frac{\text{Log}(\gamma_i)}{2\pi\sqrt{-1}}, \dots, \sum_{i=1}^n b_{i,d} \frac{\text{Log}(\gamma_i)}{2\pi\sqrt{-1}}, \frac{m_1 \text{Log}(s_1) - \text{Log}(s_1^{m_1})}{2\pi\sqrt{-1}}, \dots, \frac{m_r \text{Log}(s_r) - \text{Log}(s_r^{m_r})}{2\pi\sqrt{-1}} \right),$$

with the convention that for any $\zeta \in \mathbb{C}^\times$ with $|\zeta| = 1$, we use the principal branch of the natural logarithm, $0 \leq \frac{\text{Log} \zeta}{2\pi\sqrt{-1}} < 1$.

We first show that the way we defined v makes it an element of N . First, we have

$$(5.5) \quad \frac{1}{2\pi\sqrt{-1}} \sum_{i=1}^n b_{i,j} \text{Log}(\gamma_i) \equiv \frac{1}{2\pi\sqrt{-1}} \sum_{i=1}^n \text{Log}(\gamma_i^{b_{i,j}}) \equiv 0 \pmod{1}.$$

We also have $(m_l \text{Log}(s_l) - \text{Log}(s_l^{m_l}))/2\pi\sqrt{-1} \in \{0, \dots, m_l - 1\} = \mathbb{Z}/m_l$. So we have $v \in N$.

Next, we show that there exist rational numbers $q_i \in [0, 1) \cap \mathbb{Q}$ such that $\bar{v} = \sum_{\rho_i \in \sigma} q_i \bar{b}_i$. Indeed,

$$\bar{v} = \left(\sum_{i=1}^n b_{i,1} \frac{\text{Log}(\gamma_i)}{2\pi\sqrt{-1}}, \dots, \sum_{i=1}^n b_{i,d} \frac{\text{Log}(\gamma_i)}{2\pi\sqrt{-1}} \right),$$

and if we set $q_i := \frac{\text{Log}(\gamma_i)}{2\pi\sqrt{-1}}$, we have

$$\bar{v} = \left(\sum_{i=1}^n b_{i,1} q_i, \dots, \sum_{i=1}^n b_{i,d} q_i \right) = \sum_{i=1}^n q_i \bar{b}_i = \sum_{\rho_i \in \sigma} q_i \bar{b}_i,$$

with the last equality due to the fact that $q_i = 0$ whenever $\rho_i \notin \sigma$. So $v \in \text{Box}(\sigma)$.

It remains to show that $g \rightarrow v$ is onto. Take any $v \in \text{Box}(\sigma)$. So we have

$$v = \left(\sum_{\rho_i \in \sigma} q_i b_{i,1}, \dots, \sum_{\rho_i \in \sigma} q_i b_{i,d}, p_1, \dots, p_r \right),$$

for some $q_i \in [0, 1) \cap \mathbb{Q}$, $1 \leq i \leq n$ and for some $p_l \in \{0, \dots, m_l - 1\} = \mathbb{Z}/m_l$, $1 \leq l \leq r$. We will show there is an element of G_σ which is sent to v under the correspondence in (5.4).

Set $\gamma_i := e^{2\pi\sqrt{-1}q_i}$ and $s_l := \left(\prod_{i=1}^n e^{-2\pi\sqrt{-1}q_i b_{i,d+l}} \right)^{1/m_l} e^{2\pi\sqrt{-1}p_l/m_l}$. We claim that $(\gamma_1, \dots, \gamma_n, s_1, \dots, s_r)$ is the element of G_σ such that $g \leftrightarrow v$.

Indeed, for $1 \leq j \leq d$ we have

$$\prod_{\rho_i \in \sigma} \gamma_i^{b_{i,j}} = \prod_{\rho_i \in \sigma} \left(e^{2\pi\sqrt{-1}q_i} \right)^{b_{i,j}} = \exp \left(2\pi\sqrt{-1} \sum_{\rho_i \in \sigma} q_i b_{i,j} \right) = 1;$$

for $1 \leq l \leq r$ we have

$$\begin{aligned} s_l^{m_l} \prod_{\rho_i \in \sigma} \gamma_i^{b_{i,d+l}} &= \left(\left(\prod_{i=1}^n e^{-2\pi\sqrt{-1}q_i b_{i,d+l}} \right)^{1/m_l} e^{2\pi\sqrt{-1}p_l/m_l} \right)^{m_l} \prod_{\rho_i \in \sigma} \gamma_i^{b_{i,d+l}} \\ &= \left(\prod_{i=1}^n e^{-2\pi\sqrt{-1}q_i b_{i,d+l}} \right) e^{2\pi\sqrt{-1}p_l} \prod_{\rho_i \in \sigma} e^{2\pi\sqrt{-1}q_i b_{i,d+l}} \\ &= e^{2\pi\sqrt{-1}p_l} \\ &= 1; \end{aligned}$$

and for $\rho_i \notin \sigma$ we clearly have $q_i = 0$ and thus $\gamma_i = 1$.

■

Remark 5.5. Along with this statement, we have that every non-zero box element corresponds with a twisted sector in $I_G Z$.

Example 5.6. Recall the diagram in Figure 1 which illustrates the stacky fan for the toric stack $\mathcal{X} = \mathbb{P}(6, 4)$. By Proposition 5.4, we have the following correspondences

under (5.4):

$$\begin{array}{ll}
\text{Box}(\Sigma) & G \subset (\mathbb{C}^\times)^3 \\
v_0 = (0, 0) & \leftrightarrow g_0 = (1, 1, 1) \\
v_1 = (1, 0) & \leftrightarrow g_1 = (-1, 1, e^{\pi\sqrt{-1}/2}) \\
v_2 = (1, 1) & \leftrightarrow g_2 = (-1, 1, e^{3\pi\sqrt{-1}/2}) \\
v_3 = (0, 1) & \leftrightarrow g_3 = (1, 1, -1) \\
v_4 = (-1, 0) & \leftrightarrow g_4 = (1, e^{2\pi\sqrt{-1}/3}, 1) \\
v_5 = (-1, 1) & \leftrightarrow g_5 = (1, e^{2\pi\sqrt{-1}/3}, -1) \\
v_6 = (-2, 0) & \leftrightarrow g_6 = (1, e^{4\pi\sqrt{-1}/3}, 1) \\
v_7 = (-2, 1) & \leftrightarrow g_7 = (1, e^{4\pi\sqrt{-1}/3}, -1)
\end{array}$$

Note that Σ contains two one-dimensional cones and one zero-dimensional cone, and by definition

$$\begin{aligned}
\text{Box}(\Sigma) &= \text{Box}(\sigma_1) \bigcup \text{Box}(\sigma_2) \bigcup \text{Box}(\mathbf{0}) \\
&= \{v_0, v_1, v_2, v_3\} \bigcup \{v_0, v_3, v_4, v_5, v_6, v_7\} \bigcup \{v_0, v_3\}.
\end{aligned}$$

Remark 5.7. The construction of $\mathcal{X}(\Sigma)$ as a quotient $[Z/G]$ implies that if $v_1, v_2 \in \text{Box}(\Sigma)$ correspond to elements $g_1, g_2 \in G$ then $Z^{g_1, g_2} \neq \emptyset$ if and only if v_1, v_2 lie in a common cone of the fan Σ . In particular we can index the double inertia by pairs $\{(v_1, v_2) | v_1 \text{ and } v_2 \text{ lie in a common cone of } \sigma\}$. We refer to this subset of $\text{Box}(\Sigma) \times \text{Box}(\Sigma)$ as the *double box* and denotes it by $\text{Box}^2(\Sigma)$.

5.2. The Chow group of the inertia as a module for the stacky Stanley-Reisner ring. All inertial products are defined on the SR_Σ -module $A^*(I\mathcal{X}(\Sigma))$. (Recall that SR_Σ is the integral Chow ring of the toric stack $\mathcal{X}(\Sigma)$.) Before describing the multiplication for the inertial Chow rings we recall the module structure. We already know that components of the inertia stack are indexed by box elements. If we express $\mathcal{X}(\Sigma)$ as a quotient stack $[Z/G]$ then if $v \in \text{Box}(\Sigma)$ corresponds to $g \in G$ then the component of $I\mathcal{X}(\Sigma)$ corresponding to v is the quotient stack $[Z^g/G] = \mathcal{X}(\Sigma/\sigma(\bar{v}))$ where $\sigma(\bar{v})$ is the minimal cone of Σ containing \bar{v} [BCS, Lemma 4.6]. The Chow group of this component is a cyclic $A^*([Z/G])$ -module and we can describe this module structure using a combination of toric and equivariant methods.

Proposition 5.8. *If v is a box element then there is an isomorphism of SR_Σ -modules $A^*(\mathcal{X}(\Sigma/\sigma(\bar{v}))) \simeq \text{SR}_\Sigma / I_{\Sigma, v}$ where $I_{\Sigma, v}$ is the ideal of $\text{SR}(\Sigma)$ defined by the relations*

$$\{\tilde{x}_{i_1}, \dots, \tilde{x}_{i_k} : \rho_{i_1}, \dots, \rho_{i_k} \text{ do not lie in a cone of } \text{Star}(\sigma(\bar{v}))\}.$$

where $\text{Star} \sigma(\bar{v})$ refers to the cones of Σ containing $\sigma(\bar{v})$.

Proof. The stack $\mathcal{X}(\Sigma) = [Z/G]$ where $Z = \mathbb{A}^d \setminus V(J_\Sigma)$ where $J_\Sigma := \left\langle \prod_{\rho_i \not\subset \sigma} z_i \mid \sigma \in \Sigma \right\rangle$ is the irrelevant ideal. Now if $g \in G$ corresponds to a box element v with minimal cone

$\sigma(\bar{v})$ then $Z^g = V(z_{i_1, \dots, i_k})$ where $\rho_{i_1}, \dots, \rho_{i_k}$ are the rays of $\sigma(\bar{v})$. The proposition now follows from Proposition 2.5. ■

6. TORIC COMPUTATIONS

Let $\Sigma = (N, \Sigma, \beta)$ be a stacky fan with associated toric stack $\mathcal{X} = \mathcal{X}(\Sigma) = [Z/G]$, as constructed in Section 4. Let $g = (\gamma_1, \dots, \gamma_n, s_1, \dots, s_r) \in G$ act on Z by the diagonal action of the first n components, as defined in Section 4.

Let $\mathcal{L}_1, \dots, \mathcal{L}_n$ be the line bundles associated to the divisors where $z_i = 0$ as in Section 4.2. Writing $\mathcal{X}(\Sigma) = [Z/G]$ as above then $\mathbb{T}Z = \mathcal{L}_1 + \dots + \mathcal{L}_n$ and the vector bundle $\mathbb{T}\mathcal{X}$ corresponding to the tangent bundle of \mathcal{X} fits into an exact sequence of equivariant vector bundles [EG2, Lemma A.1]

$$0 \rightarrow T\mathcal{X} \rightarrow TZ \rightarrow \text{Lie}(G)$$

Since G is diagonalizable $\text{Lie}(G)$ is the trivial bundle so $T\mathcal{X}$ can be written in K -theory as $\sum_{i=1}^n \mathcal{L}_i - m\mathbf{1}$ where $\mathbf{1}$ is the class of the trivial one-dimensional representation of G and m is a non-negative integer. A similar analysis shows that the class corresponding to $I\mathcal{X}(\Sigma)$ can be written on each component of $I\mathcal{X}(\Sigma)$ as a sum of a subset of the \mathcal{L}_i minus a positive multiple of $\mathbf{1}$. Since logarithmic trace of the trivial bundle $\mathbf{1}$ is 0, we can make computations as if $\mathbb{T}\mathcal{X} = \mathbb{T}Z = \sum_{i=1}^n \mathcal{L}_i$.

The following Lemma tells us which of the \mathcal{L}_i appear in the expression for $\mathbb{T}I\mathcal{X}(\Sigma)$ on each component.

Lemma 6.1. *Let \mathcal{X} be the stack $\mathcal{X}(\Sigma) = [Z/G]$ associated to the stacky fan $\Sigma = (N, \Sigma, \beta)$. Let $I\mathcal{X} = [I_G Z/G]$ be the inertia stack, and $\mathbb{T}_{I\mathcal{X}}$ be the class in $K_G(I_G Z)$ corresponding to the tangent bundle. Then for any i , $1 \leq i \leq n$, and for any box element $v \in \text{Box}(\Sigma)$ and its corresponding $g = (\gamma_1, \dots, \gamma_n, s_1, \dots, s_r) \in G$, the following statements are equivalent:*

- (1) $q_i \neq 0$
- (2) $\rho_i \subseteq \sigma(\bar{v})$
- (3) $\gamma_i \neq 1$
- (4) $z_i = 0$ for all $z = (z_1, \dots, z_n) \in Z^g$
- (5) \mathcal{L}_i has coefficient 0 in $(\mathbb{T}_{I\mathcal{X}})|_{Z^g}$

Proof. Lemma 4.6 of [BCS] shows that the first four statements are equivalent when Σ is a complete simplicial fan, so it will suffice for us to show that the fifth is equivalent to the fourth. The normal bundle to the map $I_G Z \rightarrow Z$ is $\sum \mathcal{L}_i$, where the sum is over all i such that $z_i = 0$ for all $z \in Z^g$. (Note that the map $I_G X \rightarrow X$ is a closed embedding on each component of $I_G Z$.) Thus, the restriction of the tangent bundle is $(\mathbb{T}_{I_G Z})|_{Z^g} = \sum \mathcal{L}_i$, where the sum is over all i such $z_i \neq 0$ for at least one $z \in Z^g$. That is, the coefficient on \mathcal{L}_i is zero if and only if $z_i = 0$ for all $z \in Z^g$. ■

Example 6.2. Consider the stack $\mathcal{X} = \mathbb{P}(6, 4)$ of Example 5.6. We have $\overline{v_5} = -1 = 0\overline{b_1} + \frac{1}{3}\overline{b_2}$, so $q_{5,1} = 0$ and $q_{5,2} \neq 0$ (where the notation $q_{j,i}$ describes the rational coefficient for $\overline{v_j}$ on b_i). That is to say, $\sigma(\overline{v_5})$ contains ρ_2 and not ρ_1 . The corresponding group element $g_5 = (1, e^{2\pi\sqrt{-1}/3}, -1) \in G$ indeed has $\gamma_1 = 1$ and $\gamma_2 \neq 1$. Thus $Z^{g_5} = \{(z_1, 0) \mid z_1 \neq 0\}$. Finally, the restriction of the tangent bundle here is just \mathcal{L}_1 . In short, the five conditions of Lemma 6.1 are true for $i = 2$ and false for $i = 1$.

On the other hand, if we instead take the box element $v_2 = (1, 1)$ corresponding to $g_2 = (-1, 1, e^{3\pi\sqrt{-1}/2})$, the five conditions of Lemma 6.1 are true for $i = 1$ and false for $i = 2$; accordingly, the restriction of the tangent bundle is \mathcal{L}_2 .

Lastly, if we take the box element $v_3 = (0, 1)$ corresponding to $g_3 = (1, 1, -1)$, we have all five conditions false for both $i = 1$ and $i = 2$, and the restriction of the tangent bundle is $\mathcal{L}_1 + \mathcal{L}_2$. This last example illustrates an important type of box element, one in which $q_{j,i} = 0$ for all i . These correspond to elements $g \in G$ which fix all of Z .

Corollary 6.3. *If $e_1 : \mathbb{T}^2\mathcal{X} \rightarrow \mathbb{T}\mathcal{X}$ and $e_2 : \mathbb{T}^2\mathcal{X} \rightarrow \mathbb{T}\mathcal{X}$ are the evaluation maps and $(v_1, v_2) \in \text{Box}(\Sigma) \times \text{Box}(\Sigma)$, then the coefficient of \mathcal{L}_i in $(e_1^*\mathbb{T}_{I_G Z})|_{Z^{g_1, g_2}}$ (resp. $(e_2^*\mathbb{T}_{I_G Z})|_{x^{g_1, g_2}}$) is 1 if and only if $q_{1,i} = 0$ (resp. $q_{2,i} = 0$).*

As a corollary we can also describe which \mathcal{L}_i appear in the expression for $\mathbb{T}^2\mathcal{X}$

Corollary 6.4. *The coefficient of \mathcal{L}_i in $\mathbb{T}_{I_G^2 Z}$ associated to $(v_1, v_2) \in \text{Box}(\Sigma) \times \text{Box}(\Sigma)$ is 1 if and only if $q_{1,i} = q_{2,i} = 0$.*

Proof. Adapting Lemma 6.1, we have that a point $(z, g_1, g_2) \in \mathbb{I}_G^2 X$ is in the fixed locus of both $g_1 = (\gamma_{1,1}, \dots, \gamma_{1,n}, s_{1,1}, \dots, s_{1,r})$ and $g_2 = (\gamma_{2,1}, \dots, \gamma_{2,n}, s_{2,1}, \dots, s_{2,r})$ if and only if

$$\begin{aligned} z &\in \{(z_1, \dots, z_n) \in Z \mid z_i = 0 \text{ if } \gamma_{1,i} \neq 1 \text{ or } \gamma_{2,i} \neq 1\} \\ &= \{(z_1, \dots, z_n) \in Z \mid z_i = 0 \text{ if } q_{1,i} \neq 0 \text{ or } q_{2,i} \neq 0\} \end{aligned}$$

The first presentation of this set indicates that the tangent bundle to this set at (z, g_1, g_2) is

$$\mathbb{T}_{I_G^2 Z} = \sum_{\gamma_{1,i} = \gamma_{2,i} = 1} \mathcal{L}_i.$$

The second presentation indicates that the indexing set on this sum is equivalent to the set of i where $q_{1,i} = q_{2,i} = 0$. ■

Example 6.5. Take our usual example, $\mathcal{X} = \mathbb{P}(6, 4)$, in the notation as in Example 5.6. Then for the pair $(v_4, v_3) \in \text{Box}(\Sigma) \times \text{Box}(\Sigma)$, we have $e_1^*\mathbb{T}_{I_G Z} = \mathcal{L}_2$ and $e_2^*\mathbb{T}_{I_G Z} = \mathcal{L}_1 + \mathcal{L}_2$ by Corollary 6.3, and $\mathbb{T}_{I_G^2 Z} = \mathcal{L}_2$ by Corollary 6.4.

Lemma 6.6. *Let v be the box element corresponding to some $g \in G$, where $\overline{v} = \sum_{i=1}^n q_i \overline{b_i} \in \text{Box}(\Sigma)$ for some $0 \leq q_i < 1$, $i = 1, 2, \dots, n$. Then for all $i \in \{1, \dots, n\}$,*

$$L(g)(\mathcal{L}_i) = q_i \mathcal{L}_i,$$

where $L(g)$ is the logarithmic trace of Definition 3.7.

Proof. For the usual action of $g = (\gamma_1, \dots, \gamma_n, s_1, \dots, s_r)$

$$g \cdot (z_1, \dots, z_n) = (\gamma_1 z_1, \dots, \gamma_n z_n),$$

the eigenvalue on the bundle \mathcal{L}_i is γ_i . By Proposition 5.4, we have $\gamma_i = e^{2\pi\sqrt{-1}q_i}$. Thus $q_i \mathcal{L}_i$ is the logarithmic trace. ■

Example 6.7. Once again recycling Example 5.6, we have $L(g_2)(\mathcal{L}_1) = \frac{1}{2}\mathcal{L}_1$ and $L(g_2)(\mathcal{L}_2) = 0$. Similarly, $L(g_7)(\mathcal{L}_1) = 0$ and $L(g_7)(\mathcal{L}_2) = \frac{2}{3}\mathcal{L}_2$. We also have $L(g_3)(\mathcal{L}_1) = L(g_3)(\mathcal{L}_2) = 0$, which nicely illustrates the fact that $L(g)$ depends on \bar{v} but not on v . That is to say, logarithmic trace ignores torsion.

We now compute the classes \mathcal{L}_i^+ and \mathcal{L}_i^- in $K_G(\mathbb{I}_G^2 Z)$ of Proposition 3.12.

Lemma 6.8. *Let \mathcal{L}_i be the G -equivariant line bundle associated to the i -th component of the action of G on Z . For $j = 1, 2$, let $g_j = (\gamma_{j,1}, \dots, \gamma_{j,n}, s_{j,1}, \dots, s_{j,r})$ be an element of G , and let v_j be its corresponding element in $\text{Box}(\Sigma)$, where $q_{j,i}$ is defined by $\bar{v}_j = \sum_{i=1}^n q_{j,i} \bar{b}_{j,i}$ (for $1 \leq i \leq n$ and $j = 1, 2$).*

Then for each i , $1 \leq i \leq n$, exactly one of the three cases holds:

- (1) (a) At least one of $q_{1,i}, q_{2,i}$ is zero, and
 - (b) $\mathcal{L}_i^+(g_1, g_2) = 0$, and
 - (c) $\mathcal{L}_i^-(g_1, g_2) = 0$.
- (2) (a) $q_{1,i} + q_{2,i} < 1$ with $q_{1,i}, q_{2,i}$ both nonzero, and
 - (b) $\mathcal{L}_i^+(g_1, g_2) = 0$, and
 - (c) $\mathcal{L}_i^-(g_1, g_2) = \mathcal{L}_i$.
- (3) (a) $q_{1,i} + q_{2,i} \geq 1$, and
 - (b) $\mathcal{L}_i^+(g_1, g_2) = \mathcal{L}_i$, and
 - (c) $\mathcal{L}_i^-(g_1, g_2) = 0$.

Proof. Since $0 \leq q_{i,j} < 1$ for any i and j , the cases 1(a), 2(a) and 3(a) are disjoint and encompass all possibilities. So we must only show that in each of the three cases, (a) implies both (b) and (c).

For case 1, without loss of generality we may assume $q_{1,i} = 0$. Then by Lemma 6.6, $\mathcal{L}_i(g_1) = \mathcal{L}_i(g_1^{-1}) = 0$, and thus we have

$$\mathcal{L}_i^+(g_1, g_2) = L(g_2)(\mathcal{L}_i|_{Z^{g_1, g_2}}) - L(g_1 g_2)(\mathcal{L}_i|_{Z^{g_1, g_2}})$$

and

$$\mathcal{L}_i^-(g_1, g_2) = L(g_2^{-1})(\mathcal{L}_i|_{Z^{g_1, g_2}}) - L(g_2^{-1} g_1^{-1})(\mathcal{L}_i|_{Z^{g_1, g_2}}).$$

Again by Lemma 6.6, $L(g_1 g_2)(\mathcal{L}_i) = q_{2,i} \mathcal{L}_i = L(g_2)(\mathcal{L}_i)$ and $L(g_2^{-1} g_1^{-1}) = (1 - q_{2,i}) \mathcal{L}_i = L(g_2^{-1})(\mathcal{L}_i)$, which completes the proof of case 1.

For cases 2 and 3, we have

$$L(g_1 g_2)(\mathcal{L}_i) = \begin{cases} (q_{1,i} + q_{2,i})\mathcal{L}_i & \text{if } 0 < q_{1,i} + q_{2,i} < 1 \\ (q_{1,i} + q_{2,i} - 1)\mathcal{L}_i & \text{if } q_{1,i} + q_{2,i} \geq 1 \end{cases}$$

and

$$L(g_2^{-1} g_1^{-1})(\mathcal{L}_i) = \begin{cases} (1 - q_{1,i} - q_{2,i})\mathcal{L}_i & \text{if } 0 < q_{1,i} + q_{2,i} \leq 1 \\ (2 - q_{1,i} - q_{2,i})\mathcal{L}_i & \text{if } q_{1,i} + q_{2,i} > 1 \end{cases}.$$

Thus,

$$\begin{aligned} \mathcal{L}_i^+(g_1, g_2) &= L(g_1)(\mathcal{L}_i|_{Z^{g_1, g_2}}) + L(g_2)(\mathcal{L}_i|_{Z^{g_1, g_2}}) - L(g_1 g_2)(\mathcal{L}_i|_{Z^{g_1, g_2}}) \\ &= \begin{cases} 0 & \text{if } 0 < q_{1,i} + q_{2,i} < 1 \\ \mathcal{L}_i & \text{if } q_{1,i} + q_{2,i} \geq 1 \end{cases} \end{aligned}$$

and

$$\begin{aligned} \mathcal{L}_i^-(g_1, g_2) &= L(g_1^{-1})(\mathcal{L}_i|_{Z^{g_1, g_2}}) + L(g_2^{-1})(\mathcal{L}_i|_{Z^{g_1, g_2}}) - L(g_2^{-1} g_1^{-1})(\mathcal{L}_i|_{Z^{g_1, g_2}}) \\ &= \begin{cases} \mathcal{L}_i & \text{if } 0 < q_{1,i} + q_{2,i} < 1 \\ 0 & \text{if } q_{1,i} + q_{2,i} \geq 1 \end{cases}. \end{aligned}$$

■

Definition 6.9. For any pair $v_1, v_2 \in \text{Box}(\Sigma)$, define the indexing sets $B_\Sigma^+(v_1, v_2)$ and $B_\Sigma^-(v_1, v_2)$ to be the following subsets of $\{1, 2, \dots, n\}$:

$$B_\Sigma^+(v_1, v_2) := \{i \mid q_{1,i} + q_{2,i} \geq 1\}$$

$$B_\Sigma^-(v_1, v_2) := \{i \mid q_{1,i} + q_{2,i} < 1 \text{ and } q_{1,i}, q_{2,i} \neq 0\}.$$

Proposition 6.10. Let V be the G -equivariant Z -bundle $V = \sum_{i=1}^n a_i \mathcal{L}_i$, where a_i is a non-negative integer for each i . Then for any pair $v_1, v_2 \in \text{Box}(\Sigma)$,

$$V^+(g_1, g_2) = \sum_{i=1}^n a_i \mathcal{L}_i^+(g_1, g_2) = \sum_{i \in B_\Sigma^+(v_1, v_2)} a_i \mathcal{L}_i$$

and

$$V^-(g_1, g_2) = \sum_{i=1}^n a_i \mathcal{L}_i^-(g_1, g_2) = \sum_{i \in B_\Sigma^-(v_1, v_2)} a_i \mathcal{L}_i.$$

Proof. Since the logarithmic trace of a vector bundle is additive, if $V = \sum_{i=1}^n a_i \mathcal{L}_i$ then for any $g \in G$,

$$L(g)(V) = L(g) \left(\sum_{i=1}^n a_i \mathcal{L}_i \right) = \sum_{i=1}^n L(g)(a_i \mathcal{L}_i) = \sum_{i=1}^n a_i L(g)(\mathcal{L}_i).$$

The statement then follows directly from Lemma 6.8. ■

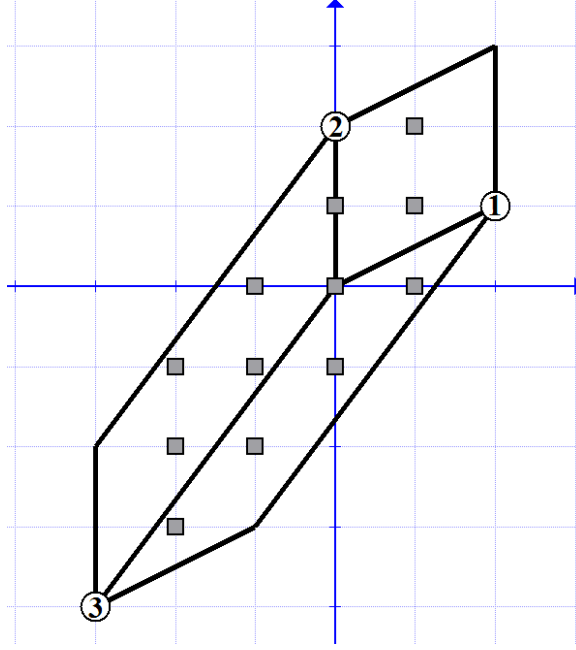


FIGURE 2. The box diagram for the stacky fan in Example 6.11 which produces $\mathbb{P}(6, 5, 4)$.

Example 6.11. The definitions of B_{Σ}^{+} and B_{Σ}^{-} may not seem particularly intuitive until we look at an example which includes a box diagram.

Consider the stacky fan Σ with $N = \mathbb{Z}^2$, with the rays of Σ generated by $b_1 = (2, 1)$, $b_2 = (0, 2)$ and $b_3 = (-3, -4)$. Figure 6 displays $\text{Box}(\Sigma)$, in a way such that the elements in $\text{Box}(\Sigma)$ (denoted by gray squares) are “boxed off” by the parallelograms formed by the b_i . Any element of $N = \mathbb{Z}^2$ outside the parallelograms is equivalent (modulo some $N(\sigma)$) to a box element lying inside one of the three parallelograms.

Consider the cone σ_1 formed by ρ_1 and ρ_2 . We’ll name the three non-zero box elements $v_1 = (0, 1)$, $v_2 = (1, 1)$ and $v_3 = (1, 2)$. Then $\bar{v}_1 = \frac{1}{2}\bar{b}_2$, $\bar{v}_2 = \frac{1}{2}\bar{b}_1 + \frac{1}{4}\bar{b}_2$ and $\bar{v}_3 = \frac{1}{2}\bar{b}_1 + \frac{3}{4}\bar{b}_2$.

The intuitive way to think about B_{Σ}^{+} is that it lists the “directions” in which the sum of the box elements “leaves” these parallelograms. For instance, with the pair v_1 and v_2 , $B_{\Sigma}^{+}(v_1, v_2) = \emptyset$, because $(0, 1) + (1, 1) = (1, 2)$ which is still inside the parallelogram. On the other hand, $B_{\Sigma}^{+}(v_2, v_3) = \{1, 2\}$, since $(1, 1) + (1, 2) = (2, 3)$, which leaves the parallelogram in the direction of both b_1 and b_2 . Similarly, $B_{\Sigma}^{+}(v_2, v_2) = \{1\}$ because the sum $v_2 + v_2$ leaves the box only in the direction of b_1 .

The other side of the coin is B_{Σ}^{-} , which lists the ways in which the sum does *not* leave the parallelogram. For example, we have $B_{\Sigma}^{-}(v_2, v_3) = \emptyset$ and $B_{\Sigma}^{-}(v_2, v_2) = \{2\}$.

An oddity occurs when one of the box elements happens to be on a ray, such as $B_{\Sigma}^-(v_1, v_2) = \{2\}$; the index 1 is not included because v_1 is on the ray b_2 (and therefore $q_{1,1} = 0$).

Let $V = a_1\mathcal{L}_1 + a_2\mathcal{L}_2 + a_3\mathcal{L}_3$ for some non-negative integers a_1, a_2, a_3 . Then we can compute the following, using Proposition 6.10 in conjunction with the above calculations:

$$\begin{aligned} V^+(v_1, v_2) &= 0 & V^-(v_1, v_2) &= a_2\mathcal{L}_2 \\ V^+(v_2, v_2) &= a_1\mathcal{L}_1 & V^-(v_2, v_2) &= a_2\mathcal{L}_2 \\ V^+(v_2, v_3) &= a_1\mathcal{L}_1 + a_2\mathcal{L}_2 & V^-(v_2, v_3) &= 0 \end{aligned}$$

7. PRESENTATIONS FOR INERTIAL CHOW RINGS

Let $\mathcal{X}(\Sigma)$ be a toric stack with stacky fan $\Sigma = (N, \Sigma, \beta)$, and let b_1, \dots, b_n be the distinguished points in N defined by β . Let $\text{Box}(\Sigma) = \{v_1, \dots, v_k\}$.

Let SR_{Σ} be the stacky Stanley-Reisner ring of the stacky fan, which equals the integral Chow ring of the stack $\mathcal{X}(\Sigma)$. The following is an immediate consequence of Proposition 5.8

Proposition 7.1. *Let $\mathcal{X}(\Sigma)$ be a toric Deligne-Mumford stack associated to the stacky fan Σ , then for any inertial product we have the following isomorphism of SR_{Σ} -modules:*

$$A^*(\mathcal{X}(\Sigma), \star, \mathbb{Z}) \cong \bigoplus_{v \in \text{Box}(\Sigma)} \text{SR}_{\Sigma} / I_{\Sigma, v}$$

where $I_{\Sigma, v}$ is the ideal defined in the statement of Proposition 5.8.

We now work to define the algebra structure on $A^*(\mathcal{X}(\Sigma), \star, \mathbb{Z})$. We begin by defining the ring R_{Σ} as the quotient $\text{SR}_{\Sigma}[\{y^v\}_{v \in \text{Box}(\Sigma)}] / \sum_v I_{\Sigma, v} y^v$. Every inertial Chow ring will be a quotient of R_{Σ} .

Proposition 7.2. *Let v_1 and v_2 be elements of $\text{Box}(\Sigma)$. If \star is any inertial product then $y^{v_1} \star y^{v_2} = 0$ if v_1, v_2 do not lie in a common cone.*

Proof. Writing $\mathcal{X}(\Sigma) = [Z/G]$ then v_1, v_2 correspond to elements $g_1, g_2 \in G$ such that $Z^{g_1}, Z^{g_2} \neq \emptyset$. However $Z^{g_1, g_2} = \emptyset$ if v_1, v_2 do not lie in a common cone. Thus $e_1^* y^{v_1} = e_2^* y^{v_2} = 0$ so $y^{v_1} \star y^{v_2} = 0$ ■

Thanks to Proposition 7.2 we make the following definition.

Definition 7.3. We define the *cone relations ideal* in R_{Σ} to be

$$\text{CR}(\Sigma) := \langle y^{v_i} y^{v_j} \mid v_i, v_j \in \text{Box}(\Sigma) \text{ and no cone contains both } v_i \text{ and } v_j \rangle.$$

Proposition 7.4. *Let $\mathcal{X}(\Sigma)$ be a toric stack and let $v_1, v_2 \in \text{Box}(\Sigma)$ such that v_1, v_2 lie in a common cone σ . Let v_3 be the box element representing $v_1 + v_2$. and corresponding to $g_3 \in G$. If \star is any inertial product then there exists a class $\text{Tw}(\star)(v_1, v_2) \in A_G^*(Z^{g_3})$ such that*

$$(7.1) \quad y^{v_1} \star y^{v_2} = y^{v_3} \cdot \text{Tw}(\star)(v_1, v_2) \cdot \prod_{q_{1,i}+q_{2,i}=1} c_1(\mathcal{L}_i),$$

Proof. With notation that we have established, the class $\prod_{q_{1,i}+q_{2,i}=1} c_1(\mathcal{L}_i)$ is the class of $\mu_*([Z^{g_1, g_2}])$ in $A^*(Z^{g_3})$. Moreover, $A^*(Z^{g_1, g_2})$ is generated as an algebra by $A^*(Z^{g_3})$. Hence by the projection formula

$$\mu_*(e_1^* y^{v_1} \cdot e_2^*(y^{v_2}) \cdot c) = y^{v_3} \cdot \text{Tw}(\star)(v_1, v_2) \cdot \prod_{q_{1,i}+q_{2,i}=1} c_1(\mathcal{L}_i),$$

for some class $\text{Tw}(\star)(v_1, v_2) \in A^*(Z^{g_3})$ whose restriction to $A^*(Z^{g_1, g_2})$ is c . ■

Definition 7.5. Let \star be an inertial product on the toric stack $\mathcal{X} = \mathcal{X}(\Sigma)$ associated to the stacky fan $\Sigma = (N, \Sigma, \beta)$. The *twisting function for the inertial product \star* is the mapping

$$\text{Tw}(\star) : \text{Box}^2(\Sigma) \rightarrow A^*(\mathbb{I}\mathcal{X})$$

which sends a box pair (v_1, v_2) to the class completely determined by \star in $A^*(\mathbb{I}\mathcal{X})$. (Recall - Remark 5.7 - that $\text{Box}^2(\Sigma)$ is the set of pairs (v_1, v_2) such that v_1, v_2 lie in a common cone.) Its component at $g_3 = g_1 g_2$, where g_1 and g_2 are the elements of G corresponding to v_1 and v_2 is given as follows:

$$(7.2) \quad y^{v_1} \star y^{v_2} = y^{v_3} \cdot \text{Tw}(\star)(v_1, v_2) \cdot \prod_{q_{1,i}+q_{2,i}=1} c_1(\mathcal{L}_i),$$

where v_3 is the unique box element such that $v_1 + v_2 = v_3$ under box addition.

As a first example, we'll compute the twisting function for the orbifold product \star_{orb} of Definition 3.15.

Example 7.6. By Definition 3.15, the orbifold product \star_{orb} is given by \star_c where $c = \text{eu}(LR(\mathbb{T})) = \text{eu}(LR(\mathbb{T}))$. Then for any pair $(v_1, v_2) \in \text{Box}^2(\Sigma)$ with corresponding pair $g_1, g_2 \in G$, let $v_3 = v_1 + v_2$ under box addition. Then we have

$$\begin{aligned} y^{v_1} \star_{orb} y^{v_2} &= y^{v_3} \cdot \text{Tw}(\star_c)(v_1, v_2) \cdot \prod_{q_{1,i}+q_{2,i}=1} c_1(\mathcal{L}_i) \\ &= y^{v_3} \cdot \text{eu}(LR(\mathbb{T}))(g_1, g_2, (g_1 g_2)^{-1}) \prod_{q_{1,i}+q_{2,i}=1} c_1(\mathcal{L}_i) \\ &= y^{v_3} \cdot \prod_{q_{1,i}+q_{2,i}>1} c_1(\mathcal{L}_i) \prod_{q_{1,i}+q_{2,i}=1} c_1(\mathcal{L}_i) \end{aligned}$$

$$= y^{v_3} \cdot \prod_{q_{1,i}+q_{2,i} \geq 1} c_1(\mathcal{L}_i)$$

Thus, by (7.2) the twisting function for the orbifold product is:

$$\text{Tw}(\star_{orb})(v_1, v_2) = \prod_{q_{1,i}+q_{2,i} > 1} c_1(\mathcal{L}_i) = \prod_{q_{1,i}+q_{2,i} > 1} \tilde{x}_i.$$

The purpose of the twisting function, illustrated in this example, is to give a succinct way of describing exactly how c affects the inertial product. We'll use this in the next definition as we continue our quest to provide a ring presentation for the inertial Chow ring.

Definition 7.7. Define $\text{BR}(\star, \Sigma)$ to be the ideal

$$\left\langle y^{v_1} y^{v_2} - y^{v_3} \cdot \text{Tw}(\star)(v_1, v_2) \cdot \prod_{q_{1,i}+q_{2,i}=1} \tilde{x}_i \mid v_1, v_2 \in \text{Box}^2(\Sigma) \right\rangle.$$

Theorem 7.8. For any toric stack $\mathcal{X}(\Sigma)$ and any inertial product \star of the form \star_{V+} or \star_{V-} for some G -equivariant vector bundle V , we have an isomorphism of \mathbb{Z} -graded rings

$$A^*(\mathcal{X}(\Sigma), \star, \mathbb{Z}) \cong \frac{R_\Sigma}{\text{CR}(\Sigma) + \text{BR}(\star, \Sigma)},$$

where the isomorphism is given by $y^{b_i} \mapsto c_1(\mathcal{L}_i)$.

Proof. This follows from Proposition 7.2 and Proposition 7.4. ■

We now compute the BR ideal for some of the inertial products defined in [EJK2].

Proposition 7.9. Let Σ be a stacky fan. For any G -equivariant bundle of the form $V = \sum a_i \mathcal{L}_i$ with $a_i \geq 0$, the box relations ideal for the \star_{V+} product is

$$\begin{aligned} \text{BR}(\star_{V+}, \Sigma) &= \left\langle y^{v_1} y^{v_2} - y^{v_3} \cdot \prod_{i \in B_\Sigma^+(v_1, v_2)} \text{eu}(\mathcal{L}_i + a_i \mathcal{L}_i) \mid (v_1, v_2) \in \text{Box}^2(\Sigma) \right\rangle \\ &= \left\langle y^{v_1} y^{v_2} - y^{v_3} \cdot \prod_{i \in B_\Sigma^+(v_1, v_2)} \tilde{x}_i^{a_i+1} \mid (v_1, v_2) \in \text{Box}^2(\Sigma) \right\rangle, \end{aligned}$$

and the box relations ideal for the \star_{V-} product is

$$\text{BR}(\star_{V-}, \Sigma) = \left\langle y^{v_1} y^{v_2} - y^{v_3} \cdot \prod_{i \in B_\Sigma^-(v_1, v_2)} \text{eu}(a_i(\mathcal{L}_i)) \cdot \prod_{j \in B_\Sigma^+(v_1, v_2)} c_1(\mathcal{L}_j) \mid (v_1, v_2) \in \text{Box}^2(\Sigma) \right\rangle$$

$$= \left\langle y^{v_1} y^{v_2} - y^{v_3} \cdot \prod_{i \in B_{\Sigma}^-(v_1, v_2)} \tilde{x}_i^{a_i} \cdot \prod_{j \in B_{\Sigma}^+(v_1, v_2)} \tilde{x}_j \middle| (v_1, v_2) \in \text{Box}^2(\Sigma) \right\rangle.$$

Proof. For any $v_1, v_2 \in \text{Box}(\sigma)$ for some cone $\sigma \in \Sigma$, we have a corresponding $g_1, g_2 \in G$ and we can write $\overline{v_1} = \sum_{\rho_i \in \sigma} q_{1,i} \overline{b_i}$ and $\overline{v_2} = \sum_{\rho_i \in \sigma} q_{2,i} \overline{b_i}$. Let $v_3 = v_1 + v_2$ under addition in $\text{Box}(\sigma)$.

We first compute the twisting function for the \star_{V+} product. To do so, we need the Euler class of \mathcal{R}^+V , using Proposition 3.12 and Proposition 6.10. We have

$$(R^+V + LR(\mathbb{T}))(g_1, g_2) = V^+(g_1, g_2) + LR(\mathbb{T})(g_1, g_2) = \sum_{i \in B_{\Sigma}^+(v_1, v_2)} a_i \mathcal{L}_i + \sum_{q_{1,j} + q_{2,j} > 1} \mathcal{L}_j,$$

and so

$$\begin{aligned} \text{Tw}(\star_{V+})(v_1, v_2) &= \text{eu} \left(\sum_{i \in B_{\Sigma}^+(v_1, v_2)} a_i \mathcal{L}_i + \sum_{q_{1,j} + q_{2,j} > 1} \mathcal{L}_j \right) \\ &= \prod_{B_{\Sigma}^+(v_1, v_2)} \text{eu}(a_i \mathcal{L}_i) \cdot \prod_{q_{1,i} + q_{2,i} > 1} c_1(\mathcal{L}_i). \end{aligned}$$

The normal bundle for $\mathbf{g} = (g_1, g_2)$ is $\sum \mathcal{L}_i$, where the sum is over all i such that $\rho_i \in (\overline{v_1}, \overline{v_2})$ but $\rho_i \notin \sigma(\overline{v_3})$. Equivalently, we could say the sum is over all i such that $q_{1,i} + q_{2,i} = 1$.

Then we have

$$\begin{aligned} y^{v_1} \star_{V+} y^{v_2} &= \mu_*(e_1^* y^{v_1} \cdot e_2^* y^{v_2} \cdot \text{eu}(\mathcal{R}^+V)) \\ &= y^{v_3} \cdot \left(\prod_{i \in B_{\Sigma}^+(v_1, v_2)} \text{eu}(a_i \mathcal{L}_i) \cdot \prod_{q_{1,j} + q_{2,j} > 1} c_1(\mathcal{L}_j) \right) \cdot \prod_{q_{1,k} + q_{2,k} = 1} c_1(\mathcal{L}_k) \\ &= y^{v_3} \cdot \prod_{i \in B_{\Sigma}^+(v_1, v_2)} \text{eu}(\mathcal{L}_i + a_i \mathcal{L}_i) \end{aligned}$$

where the last equality follows because $B_{\Sigma}^+(v_1, v_2) = \{i | q_{1,i} + q_{2,i} \geq 1\}$.

Thus,

$$\text{BR}(\star_{V+}, \Sigma) = \left\langle y^{v_1} y^{v_2} - y^{v_3} \cdot \prod_{i \in B_{\Sigma}^+(v_1, v_2)} \text{eu}(\mathcal{L}_i + a_i \mathcal{L}_i) \middle| v_1, v_2 \in \text{Box}^2(\Sigma) \right\rangle.$$

If we instead consider the \star_{V^-} product, we instead have the twisting function

$$\mathrm{Tw}(\star_{V^-})(v_1, v_2) = \mathrm{eu} \left(\sum_{i \in B_{\Sigma}^-(v_1, v_2)} a_i \mathcal{L}_i + \sum_{q_{1,j} + q_{2,j} > 1} \mathcal{L}_j \right) = \prod_{i \in B_{\Sigma}^-(v_1, v_2)} \mathrm{eu}(a_i \mathcal{L}_i) \cdot \prod_{q_{1,j} + q_{2,j} > 1} c_1(\mathcal{L}_j)$$

and so

$$\begin{aligned} w_1 \star_{V^-} w_2 &= \mu_*(e_1^* w_1 \cdot e_2^* w_2 \cdot \mathrm{eu}(\mathcal{K}^- V)) \\ &= w_3 \cdot \left(\prod_{i \in B_{\Sigma}^-(v_1, v_2)} \mathrm{eu}(a_i \mathcal{L}_i) \cdot \prod_{q_{1,j} + q_{2,j} > 1} c_1(\mathcal{L}_j) \right) \cdot \prod_{q_{1,k} + q_{2,k} = 1} c_1(\mathcal{L}_k) \\ &= w_3 \cdot \prod_{i \in B_{\Sigma}^-(v_1, v_2)} \mathrm{eu}((\mathcal{L}_i)) \cdot \prod_{q_{1,j} + q_{2,j} \geq 1} c_1(\mathcal{L}_j). \end{aligned}$$

Therefore,

$$\mathrm{BR}(\star_{V^-}, \Sigma) = \left\langle y^{v_1} y^{v_2} - y^{v_3} \cdot \prod_{i \in B_{\Sigma}^-(v_1, v_2)} \mathrm{eu}(a_i \mathcal{L}_i) \cdot \prod_{j \in B_{\Sigma}^+(v_1, v_2)} c_1(\mathcal{L}_j) \middle| v_1, v_2 \in \mathrm{Box}(\Sigma) \right\rangle.$$

■

Corollary 7.10. *For the virtual product of $[\mathrm{GLS}^+]$, we use $V = \mathbb{T} = \sum_{i=1}^n \mathcal{L}_i$ in the \star_{V^-} product. Then*

$$\mathrm{Tw}(\star_{\mathrm{virt}})(v_1, v_2) = \prod_{\rho_i \subseteq \sigma(\overline{v_1}), \sigma(\overline{v_2})} c_1(\mathcal{L}_i).$$

Example 7.11. For the toric stack $\mathcal{X}(\Sigma)$ of Example 6.11, consider $v_1 = (0, 1)$ and $v_3 = (1, 2)$.

Then $q_{1,1} + q_{3,1} = 0 + \frac{1}{2} = \frac{1}{2}$ and $q_{1,2} + q_{3,2} = \frac{1}{2} + \frac{3}{4} = \frac{5}{4}$. So under the orbifold product, we have $\mathrm{Tw}(v_1, v_3)(\star_{\mathrm{orb}}) = c_1(\mathcal{L}_2)$.

On the other hand, we have $\rho_2 \in \sigma(\overline{v_1})$ while $\rho_1, \rho_2 \in \sigma(\overline{v_3})$. So under the virtual product, we have $\mathrm{Tw}(v_1, v_3)(\star_{\mathrm{virt}}) = c_1(\mathcal{L}_2)$.

We now give an example of a complete presentation of the inertial Chow ring for an inertial product coming from a vector bundle on $\mathcal{X}(\Sigma)$.

Example 7.12. Let Σ be the stacky fan of Example 6.11, with $\mathcal{X}(\Sigma) = \mathbb{P}(6, 5, 4)$. As before, let \mathcal{L}_i be the G -equivariant line bundle associated to the i -th component of the action of G on $\mathcal{X}(\Sigma)$, and to the ray ρ_i , for $i = 1, 2, 3$. Let $V = a_1 \mathcal{L}_1 + a_2 \mathcal{L}_2 + a_3 \mathcal{L}_3$, for non-negative integers a_1, a_2, a_3 . We'll compute the Chow ring for the \star_{V^+} product on $\mathcal{X}(\Sigma)$.

As stated previously, we can calculate the ring R_{Σ} and its ideals $I(\Sigma)$, $\mathrm{CR}(\Sigma)$ and $\mathrm{Cir}(\Sigma)$ without regard to the vector bundle V , so we'll do this first.

The distinguished points are $b_1 = (2, 1)$, $b_2 = (0, 2)$ and $b_3 = (-3, -4)$, so the box elements are

$$\begin{array}{llll} v_1 = (0, 1) & v_4 = (-1, 0) & v_7 = (-2, -2) & v_{10} = (0, -1) \\ v_2 = (1, 1) & v_5 = (-1, -1) & v_8 = (-2, -3) & v_{11} = (1, 0) \\ v_3 = (1, 2) & v_6 = (-2, -1) & v_9 = (-1, -2) & \end{array}$$

with $q_{j,i}$ being defined by $\overline{v_j} = q_{j,1}\overline{b_1} + q_{j,2}\overline{b_2} + q_{j,3}\overline{b_3}$ as follows:

$$\begin{array}{llll} \overline{v_1} = \frac{1}{2}\overline{b_2} & \overline{v_4} = \frac{2}{3}\overline{b_2} + \frac{1}{3}\overline{b_3} & \overline{v_7} = \frac{1}{3}\overline{b_2} + \frac{2}{3}\overline{b_3} & \overline{v_{10}} = \frac{3}{5}\overline{b_1} + \frac{2}{5}\overline{b_3} \\ \overline{v_2} = \frac{1}{2}\overline{b_1} + \frac{1}{4}\overline{b_2} & \overline{v_5} = \frac{1}{6}\overline{b_2} + \frac{1}{3}\overline{b_3} & \overline{v_8} = \frac{1}{5}\overline{b_1} + \frac{4}{5}\overline{b_3} & \overline{v_{11}} = \frac{4}{5}\overline{b_1} + \frac{1}{5}\overline{b_3} \\ \overline{v_3} = \frac{1}{2}\overline{b_1} + \frac{3}{4}\overline{b_2} & \overline{v_6} = \frac{5}{6}\overline{b_2} + \frac{2}{3}\overline{b_3} & \overline{v_9} = \frac{2}{5}\overline{b_1} + \frac{3}{5}\overline{b_3} & \end{array}$$

along with $\mathbf{0} = (0, 0)$ (see Figure 6 for an intuition-aiding diagram). The fan Σ has three top-dimensional cones:

$$\text{Box}(\sigma_1) = \{\mathbf{0}, v_1, v_2, v_3\} \quad \text{Box}(\sigma_2) = \{\mathbf{0}, v_1, v_4, v_5, v_6, v_7\} \quad \text{Box}(\sigma_3) = \{\mathbf{0}, v_8, v_9, v_{10}, v_{11}\}$$

Let $w_j := y^{v_j}$ for $j = 1, \dots, 11$, and let $x_i := y^{b_i}$ for $i = 1, 2, 3$. Then with coefficients in \mathbb{Z} , the stacky Stanley-Reisner ring is

$$\text{SR}_\Sigma = \mathbb{Z}[x_1, x_2, x_3] / (I(\Sigma) + \text{Cir}(\Sigma)).$$

Since the orbifold is reduced we have $x_i = \tilde{x}_i$ for each i . Based solely on the fan structure, we can compute the irrelevant ideal:

$$I(\Sigma) = \langle x_1 x_2 x_3 \rangle$$

Using the distinguished points we compute the circuit ideal:

$$\text{Cir}(\Sigma) = \langle 2x_1 - 3x_3, x_1 + 2x_2 - 4x_3 \rangle$$

so

$$\text{SR}_\Sigma = \mathbb{Z}[x_1, x_2, x_3] / (2x_1 - 3x_3, x_1 + 2x_2 - 4x_3, x_1 x_2 x_3)$$

Since the box has 11 elements, the ring R_Σ is a quotient of the ring

$$\text{SR}_\Sigma[w_1, w_2, \dots, w_{10}]$$

by the ideal

$$\sum_{i=1}^{11} I_{\Sigma, v_i} = (x_1 x_3 w_1, x_3 w_2, x_3 w_2, x_1 w_4, x_1 w_5, x_1 w_6, x_1 w_7, x_2 w_8, x_2 w_9, x_2 w_{10}, x_2 w_{11}).$$

Finally, based on the boxes of the top-dimensional cones σ_1 , σ_2 and σ_3 , we have the cone relations ideal:

$$\text{CR}(\Sigma) = \left\langle \begin{array}{l} w_1 w_8, w_1 w_9, w_1 w_{10}, w_1 w_{11}, w_2 w_4, w_2 w_5, w_2 w_6, w_2 w_7, w_2 w_8, w_2 w_9, \\ w_2 w_{10}, w_2 w_{11}, w_3 w_4, w_3 w_5, w_3 w_6, w_3 w_7, w_3 w_8, w_3 w_9, w_3 w_{10}, w_3 w_{11}, \\ w_4 w_8, w_4 w_9, w_4 w_{10}, w_4 w_{11}, w_5 w_8, w_5 w_9, w_5 w_{10}, w_5 w_{11}, w_6 w_8, w_6 w_9, \\ w_6 w_{10}, w_6 w_{11}, w_7 w_8, w_7 w_9, w_7 w_{10}, w_7 w_{11} \end{array} \right\rangle$$

This leaves only box relations ideal, which depends on V . By Proposition 7.9, we have

$$\text{BR}(\star_{V^+}, \Sigma) = \left\langle w_{j_1} w_{j_2} - w_{j_3} \cdot \prod_{q_{1,i} + q_{2,i} \geq 1} c_1(\mathcal{L}_i)^{1+a_i} \middle| v_{j_1}, v_{j_2} \in \text{Box}(\Sigma) \right\rangle$$

For each of the three top-dimensional cones σ , we must compute $w_{j_1} w_{j_2}$ for every pair $(v_{j_1}, v_{j_2}) \in \text{Box}(\sigma)$. (Note that we need not calculate for box elements which are not in the same top-dimensional cone, such as $y^{v_1} y^{v_8}$, since it's already an ideal generator in $\text{CR}(\Sigma)$.)

So, the generators of $\text{BR}(\star_{V^+}, \Sigma)$, or at least the ones that we need to pay attention to, are

$$\begin{array}{ll} w_1 w_2 - w_3 & w_1 w_3 - w_2 \cdot c_1(\mathcal{L}_2)^{a_2+1} \\ w_2 w_3 - 1 \cdot c_1(\mathcal{L}_1)^{a_1+1} \cdot c_1(\mathcal{L}_2)^{a_2+1} & w_1^2 - 1 \cdot c_1(\mathcal{L}_2)^{a_2+1} \\ w_2^2 - w_1 \cdot c_1(\mathcal{L}_1)^{a_1+1} & w_3^2 - w_1 \cdot c_1(\mathcal{L}_1)^{a_1+1} \cdot c_1(\mathcal{L}_2)^{a_2+1} \\ w_1 w_4 - w_5 \cdot c_1(\mathcal{L}_2)^{a_2+1} & w_1 w_5 - w_4 \\ w_1 w_6 - w_7 \cdot c_1(\mathcal{L}_2)^{a_2+1} & w_1 w_7 - w_6 \\ w_4 w_5 - w_6 & w_4 w_6 - w_1 \cdot c_1(\mathcal{L}_2)^{a_2+1} \cdot c_1(\mathcal{L}_3)^{a_3+1} \\ w_4 w_7 - 1 \cdot c_1(\mathcal{L}_2)^{a_2+1} \cdot c_1(\mathcal{L}_3)^{a_3+1} & w_5 w_6 - 1 \cdot c_1(\mathcal{L}_2)^{a_2+1} \cdot c_1(\mathcal{L}_3)^{a_3+1} \\ w_5 w_7 - w_1 \cdot c_1(\mathcal{L}_3)^{a_3+1} & w_6 w_7 - w_5 \cdot c_1(\mathcal{L}_2)^{a_2+1} \cdot c_1(\mathcal{L}_3)^{a_3+1} \\ w_4^2 - w_7 \cdot c_1(\mathcal{L}_2)^{a_2+1} & w_5^2 - w_7 \\ w_6^2 - w_4 \cdot c_1(\mathcal{L}_2)^{a_2+1} \cdot c_1(\mathcal{L}_3)^{a_3+1} & w_7^2 - w_4 \cdot c_1(\mathcal{L}_3)^{a_3+1} \\ w_8 w_9 - w_{10} \cdot c_1(\mathcal{L}_3)^{a_3+1} & w_8 w_{10} - w_{11} \cdot c_1(\mathcal{L}_3)^{a_3+1} \\ w_8 w_{11} - 1 \cdot c_1(\mathcal{L}_1)^{a_1+1} \cdot c_1(\mathcal{L}_3)^{a_3+1} & w_9 w_{10} - 1 \cdot c_1(\mathcal{L}_1)^{a_1+1} \cdot c_1(\mathcal{L}_3)^{a_3+1} \\ w_9 w_{11} - w_8 \cdot c_1(\mathcal{L}_1)^{a_1+1} & w_{10} w_{11} - w_9 \cdot c_1(\mathcal{L}_1)^{a_1+1} \\ w_8^2 - w_9 \cdot c_1(\mathcal{L}_3)^{a_3+1} & w_9^2 - w_{11} \cdot c_1(\mathcal{L}_3)^{a_3+1} \\ w_{10}^2 - w_8 \cdot c_1(\mathcal{L}_1)^{a_1+1} & w_{11}^2 - w_{10} \cdot c_1(\mathcal{L}_1)^{a_1+1} \end{array}$$

From these generator relations, we can eliminate w_3, w_4, w_6, w_7 .

By the associated formula of Σ (see Lemma 4.4), we have the following:

$$\begin{aligned} c_1(\mathcal{L}_1) &= x_1 = \tilde{x}_1 = 6t \\ c_1(\mathcal{L}_2) &= x_2 = \tilde{x}_2 = 5t \\ c_1(\mathcal{L}_3) &= x_3 = \tilde{x}_3 = 4t \end{aligned}$$

Finally, we have our ring presentation for $A^*(\mathcal{X}(\Sigma), \star_{V^+}, \mathbb{Z})$:

$$\frac{\mathbb{Z}[t, w_1, w_2, w_5, w_8, w_9, w_{10}, w_{11}]}{120t^3, 24t^2w_1, 4tw_2, 6tw_5, 5tw_8, 5tw_9, 5tw_{10}, 5tw_{11},} \left\langle \begin{array}{l} w_1w_8, w_1w_9, w_1w_{10}, w_1w_{11}, w_2w_5, w_2w_8, w_2w_9, w_2w_{10}, w_2w_{11}, w_5w_8, w_5w_9, \\ w_5w_{10}, w_5w_{11}, w_1^2 - (5t)^{a_2+1}, w_2^2 - w_1(6t)^{a_1+1}, w_5^4 - w_1w_5(4t)^{a_3+1}, \\ w_8^2 - w_9(4t)^{a_3+1}, w_8w_9 - w_{10}(4t)^{a_3+1}, w_8w_{10} - w_{11}(4t)^{a_3+1}, \\ w_8w_{11} - (6t)^{a_1+1}(4t)^{a_3+1}, w_9^2 - w_{11}(4t)^{a_3+1}, w_9w_{10} - (6t)^{a_1+1}(4t)^{a_3+1}, \\ w_{10}^2 - w_8(6t)^{a_1+1}, w_9w_{11} - w_8(6t)^{a_1+1}, w_{11}^2 - w_{10}(6t)^{a_1+1}, w_{10}w_{11} - w_9(6t)^{a_1+1} \end{array} \right\rangle$$

8. A NEW ASYMPTOTIC PRODUCT

Given a vector bundle V on a toric stack $\mathcal{X}(\Sigma)$ we show that we can produce new associative products on $A^*(I\mathcal{X})_{\mathbb{Q}}$ by taking the limits of the \star_{aV^+} and \star_{aV^-} products where a is a positive integer going to ∞ .

Theorem 8.1. *The following formulas define associative products on $A^*(I\mathcal{X})_{\mathbb{Q}}$.*

- (1) $y^{v_1} \star_{+\infty} y^{v_2} = 0$ if $B_{\Sigma}^+(v_1, v_2)$ is nonempty, and
- (2) $y^{v_1} \star_{+\infty} y^{v_2} = y^{v_1} \star y^{v_2}$ otherwise.
- (1) $y^{v_1} \star_{-\infty} y^{v_2} = 0$ if $B_{\Sigma}^-(v_1, v_2)$ is nonempty, and
- (2) $y^{v_1} \star_{-\infty} y^{v_2} = y^{v_1} \star y^{v_2}$ otherwise.

(Here \star is the usual orbifold product.)

Proof. Let $V = \sum_{i=1}^n \mathcal{L}_i$ and let a be a positive integer. The box relations ideal of Proposition 7.9 for the aV^+ product becomes

$$\text{BR}(\star_{aV^+}, \Sigma) = \left\langle y^{v_1}y^{v_2} - y^{v_3} \cdot \prod_{i \in B_{\Sigma}^+} c_1(\mathcal{L}_i)^{1+a} \middle| v_1, v_2 \in \text{Box}(\Sigma) \right\rangle$$

Now since $A^r(I\mathcal{X})_{\mathbb{Q}} = 0$ for $r < \dim \mathcal{X}$, the terms $c_1(\mathcal{L}_i)^{1+a}$ vanish for a sufficiently large. In this case the box relation ideal becomes

$$\{y^{v_1}y^{v_2} | B_{\Sigma}^+(v_1, v_2) \neq \emptyset\} \cup \{y^{v_1}y^{v_2} - y^{v_3} | B_{\Sigma}^+(v_1, v_2) = \emptyset\}.$$

which gives $\star_{+\infty}$ formula of Theorem 8.1. The associativity of the $\star_{-\infty}$ product follows from a similar argument with the \star_{aV^-} -product. ■

Other associative products with fewer non-trivial products can be obtained by taking $V = \mathcal{L}_{i_1} + \dots + \mathcal{L}_{i_m}$ where $\{i_1, \dots, i_m\}$ is a subset of $\{1, \dots, n\}$.

Example 8.2. Consider the stacky fan Σ of Example 7.12. We'll compute the Chow ring of $\mathcal{X}(\Sigma)$ with $\star_{+\infty}$ product. Note that the calculation for the inertial Chow ring $A^*(\mathcal{X}(\Sigma), \star_{V_{\infty}^+}, \mathbb{Q})$ will be the same as with $A^*(\mathcal{X}(\Sigma), \star_{V^+}, \mathbb{Q})$, done in Example 7.12 except that all terms with a_i as exponents vanish. So, we have

$$A^*(\mathcal{X}(\Sigma), \star_{+\infty}, \mathbb{Q}) \cong \frac{\mathbb{Q}[t, w_1, w_2, w_5, w_8, w_9, w_{10}, w_{11}]}{\left\langle \begin{array}{l} t^3, t^2w_1, tw_2, tw_5, tw_8, tw_9, tw_{10}, tw_{11} \\ w_1w_8, w_1w_9, w_1w_{10}, w_1w_{11}, w_2w_5, w_2w_8, w_2w_9, w_2w_{10}, w_2w_{11}, \\ w_5w_8, w_5w_9, w_5w_{10}, w_5w_{11}, w_1^2, w_2^2, w_5^4, w_8^2, w_8w_9, w_8w_{10}, \\ w_8w_{11}, w_9^2, w_9w_{10}, w_{10}^2, w_9w_{11}, w_{11}^2, w_{10}w_{11} \end{array} \right\rangle}$$

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